A PRIORI VOTING POWER WHEN ONE VOTE COUNTS IN TWO WAYS, WITH APPLICATION TO TWO VARIANTS OF THE U.S. ELECTORAL COLLEGE

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Abstract

When we try to measure the *a priori* voting power of individual voters under proposed variants of the two-tier U.S. Electoral College system, two plans present special difficulties: the 'Modified District Plan,' under which a candidate is awarded one electoral vote for each Congressional District he carries and two electoral votes for each state he carries, and the 'National Bonus Plan,' under which a candidate is awarded all the electoral votes of each state he carries (as at present) plus a 'national bonus' of some fixed number of electoral votes if he wins the national popular vote. This difficulty arises because, under these arrangements, each voter casts a single vote that counts two ways: in the voter's district and state under the Modified District Plan, and in the voter's state and the nation as a whole under the National Bonus Plan. In his original analysis of voting power under Electoral College variants, Banzhaf (1968) evaluated voting power under the Modified District Plan by calculating a voter's two-stage voting power first through the district vote and then through the state vote and then adding the two values together. Unfortunately, this approach cannot be justified, because it ignores interdependencies in the way district and state electoral votes may be cast — in particular, even though individuals cast statistically independent votes, the fact that they are casting votes that count in the same way in two tiers induces a correlation between popular votes at different levels. That this problem is serious is indicated by the fact that mean individual voting power under the Modified District system, when calculated in the Banzhaf manner, exceeds individual voting power under direct national popular vote, which Felsenthal and Machover (1998) show is a logical impossibility for a simple voting game. While an analytic solution to this problem may be possible, the difficulties appear to be formidable, and I proceed computationally by generating a very large sample of random (or Bernoulli) elections, with electoral votes awarded to the candidates on the basis of each plan. This generates a database that can be manipulated to determine the expected distributions of electoral votes for a candidate under specified contingencies with respect to first-tier voting, from which relevant second-tier probabilities can be inferred.

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The President of the United States is elected, not by a direct national popular vote, but by a two-tier Electoral College system in which (in almost universal practice since the 1830s) separate state popular votes are aggregated by adding up state electoral votes awarded, on a winner-take-all basis, to the plurality winner in each state. Each state has electoral votes equal in number to its total representation in Congress and since 1964 the District of Columbia has three electoral votes. At the present time, there are 435 members of the House of Representative, 100 Senators, so the total number of electoral votes is 538, with 270 required for election (and a 269-269 tie possible). The U.S. Electoral College is therefore a *two-tier* electoral system: individual voters cast votes in the first tier to choose between rival slates of 'Presidential electors' pledged to one or other Presidential candidate, and the winning elector slates then cast blocs of electoral votes for the candidate to whom they are pledged in the second tier. The Electoral College therefore generates the kind of weighted voting system that invites analysis using one of the several measures of *a priori* voting power. With such a measure, we can determine whether and how much the power of voters may vary from state to state and how voting power would change under different variants of the Electoral College system.

1. Individual Voting Power under the Electoral College

Several years ago, I had a commission to write an encyclopedia entry on "Voting Power in the U.S. Electoral College" (Miller, 2011), and I decided to include a chart displaying individual voting power by state. Having been introduced some years earlier to Dan Felsenthal and Moshe Machover's magnificent treatise on *The Measurement of Voting Power*, I was confident that I had a reasonably precise understanding of the properties and proper interpretations of the various voting power measures (with which I had been broadly familiar since graduate school). I also believed that I could make the necessary calculations using the immensely useful website on *Computer Algorithms for Voting Power Analysis* created and maintained by Dennis Leech and Robert Leech.

I was persuaded by Felsenthal and Machover's emphatic advice that the (absolute) *Banzhaf measure* is the proper measure of *a priori* voting power in the context of ordinary two-candidate or two-party elections. Given *n* voters, there are 2^{n-1} *bipartitions* (i.e., complementary pairs of subsets) of voters (including the pair consisting of the set of all voters and the empty set). A voter is *critical* in a bipartition if the set to which the voter belongs is *winning* (e.g., a set of states with 270 electoral voters) but would not be winning if the voter belonged to the complementary set. A voter's *Banzhaf score* is the total number of bipartitions in which the voter is critical. A voter's *absolute Banzhaf voting power* is the voter's Banzhaf score divided by the number of bipartitions.

Felsenthal and Machover show that the absolute Banzhaf measure (unlike the 'relative' Banzhaf index or the Shapley-Shubik index) has the following directly meaningful and analytically useful probabilistic interpretation. Suppose we know nothing about individual voters except their position with respect to the formal properties of a voting system (in this case, what state they live in) but nothing about their political inclinations, voting habits, etc., and that, from behind this 'veil of ignorance,' we wish to assess their voting power. For this purpose (though certainly not for some others), our *a priori* expectation must be that individuals vote randomly, i.e., as if they are independently flipping fair coins in what may be called a *random* (or *Bernoulli*) *election*. On this assumption, Felsenthal and Machover show that a voter's absolute Banzhaf voting power is the probability that he or she casts a *decisive* vote that determines the outcome of the election (e.g., that, given all other votes, breaks what would otherwise be a tie).

Now suppose also that we know nothing about U.S. Presidential elections other than the formal rules of the Electoral College — specifically, we know the population of each state, the total number of electoral votes, the formula for apportioning these electoral votes among the states on the basis of population, and the fact that each state's electoral votes are cast as block for the candidate who wins the most popular votes in the state. Absent any further information, we must assume that the total number of votes cast in a state is equal to some fixed percent of the state's apportionment population. In a two-tier voting system such as the Electoral College, voter *i*'s *a priori* voting power is the probability that *i* casts a *doubly decisive* vote (e.g., one that breaks what would otherwise be a tie in the state popular vote in the voter's state, which in turn breaks what would otherwise be a deadlock in the Electoral College). Put otherwise, the *a priori* voting power of a voter under the existing Electoral College is:

the probability that the voter cast a decisive vote within his state

times

the probability that the state casts a decisive bloc of electoral votes in the Electoral College, given that the voter is decisive within his state.

The probability that a voter casts a decisive vote in the state is essentially the probability that the state vote is tied, which is equal (to excellent approximation given a modestly large number n of voters) to $\sqrt{2 / \pi n}$. The probability that the voter's state casts a decisive block of votes in the Electoral College is equal to the state's absolute Banzhaf power in the weighted voting game 51:538(270: 55, 34, . . . , 3), i.e., one with 51 voters, a total weight of 538, a winning quota of 270, a weight of 55 for the largest player (California), 34 for the next largest (Texas), etc. The Banzhaf value for each state can be calculated using the appropriate algorithm (namely, *ipgenf*) from the *Voting Power Analysis* website. Since Banzhaf values are equivalent to the relevant probabilities, overall two-tier voting power for any voter is the product of these two quantities. Moreover, the probability that a state casts a decisive bloc of votes in the Electoral College is not conditional on the popular vote outcome within the state, so the qualification 'given that the voter is decisive within his state' in the formulation above is unnecessary.

In this way, I could readily produce the chart displayed in Figure 1, which shows how individual voting power varies across states with different populations (based on the 2000 census). Since probabilities of individual decisiveness are very small, it is convenient to rescale voting power in this and similar charts that follow so that individual voting power in the least favored state is set at 1.0 and in other states as multiples of this. The figure also shows both mean individual voting power under the Electoral College and individual voting power under direct popular vote. The latter is of course the same for all voters and, perhaps surprisingly, it is substantially greater that mean voting power under the Electoral College — indeed, it is greater than the power of voters in every state other than the most favored California.

Having completed my encyclopedia entry, I thought it would be interesting and straightforward to make similar charts for possible variants of the Electoral College. The variants I considered fell into three categories: those that keep the state-level winner-take-all practice but use a different formula for apportioning electoral votes among states (e.g., basing electoral votes on House seats only, giving all states equal electoral votes, etc.), those that keep the existing apportionment of electoral votes but use something other than winner-take-all for the casting of state electoral votes, and a range of 'national bonus' plans.

All variants in the first category and also the Pure District Plan (under which each state is divided into as many equally populated electoral districts as it has electoral votes, and a candidate wins one electoral vote for each district carried) in the second category are simple two-tier systems, in which voting power calculations can be made in just the same way as for the existing Electoral College. The Pure Proportional Plan (under which each state's electoral votes are apportioned among the candidates in a way that is precisely proportional to their popular vote shares in that state) and the Whole Number Proportional Plan (under which each state's electoral votes state are apportioned among the candidates on the basis of their popular vote shares, but in whole numbers using an apportionment formula in the manner of proportional representation electoral systems) require somewhat different but still straightforward calculations. However, the Modified District Plan (under which a candidate wins one electoral vote for each Congressional District he carries *and* two electoral votes for each state he carries) and any National Bonus Plan (under which electoral votes are apportioned and cast as under the existing system at present but the candidate who wins the most popular votes nationwide is *also* awarded a bonus of some number additional electoral votes) present special difficulties. This is because each voter casts a single vote that counts two ways: in the voter's district and state under the Modified District Plan, and in the voter's state and the nation as a whole under the National Bonus Plan. This means that the probability that a state casts a decisive pair of votes in the Electoral College (under the Modified District Plan), or the bonus is decisive (under the National Bonus Plan) depends on whether the voter casts a decisive vote at the district or state level, respective. Thus, the stipulation 'given that the voter is decisive within his state' in the formulation of double decisiveness is now necessary, at least in principle — though one might speculate (incorrectly) that it makes almost no difference in practice.

In his original work on voting power in the Electoral College, Banzhaf (1968) attempted to calculate individual two-tier voting power under the Modified District Plan by (i) calculating individual voting power through the voter's district, (ii) separately calculating individual voting power through the voter's state, and then (iii) adding these two probabilities together. Figure 2 displays voting power under the Modified District Plan calculated in the Banzhaf manner. While the relative voting power of voters in different states appears reasonable, Figure 2 displays a major problem in that mean individual voting power (slightly) exceeds individual voting power under direct popular vote. This is a problem because Felsenthal and Machover (1998, pp. 58-59) demonstrate that, within the class of ordinary voting systems, mean individual voting power under direct popular vote maximizes the total Banzhaf score of all voters and therefore mean voting power. This anomaly was not evident in Banzhaf's original analysis, because he reported only rescaled voting power values, and he never made comparisons of absolute individual voting power across Electoral College variants or with the direct popular vote system.[1]

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[1] Recalculation of Banzhaf's results (using 1960 apportionment populations) shows that the same anomaly exists there.

More recent work by Edelman (2004) clarifies the nature of this problem but does not itself point to a solution. Edelman argues that individual voting power in two-tier voting systems of a representative nature (e.g., council or legislature) can be enhanced by providing some at-large representation as well as typical (Anglo-American) single-member district representation. Edelman further shows that if voters cast separate and independent votes for their district and at-large representatives, and if the at-large representatives are elected on a winner-take-all basis and vote as a bloc in the top tier, individual voting may be calculated by separately calculating individual voting power through the voter's district representation and through at-large representation and adding these two probabilities together (essentially as Banzhaf tried to calculate voting power under the Modified District Plan). Edelman further shows that individual voting power so calculated is maximized when the number of at-large representatives is as close as possible to the square root of the total number of representatives and that this maximum voting power exceeds individual voting power when all members are elected at-large.

The key assumption in Edelman's analysis is that voters cast *separate and independent votes* for district and at-large representation. Edelman claims that allowing separate and independent votes gives a voter more power because "he has more flexibility in the way he casts his vote." In many contexts, greater "flexibility" in casting votes may be valuable to voters, but only if possible election outcomes have multiple attributes that voters care about, e.g., if voters care about not only what party controls the council, but also about its ideological balance, ethnic diversity, geographical representation, etc. But the foundational assumption of voting power theory is that "the measurement of voting power . . . concerns any collective that makes *yes-or-no decisions* by vote" (Felsenthal and Machover, 1998, p. 1), i.e., the setup is based on votes and outcomes that are both binary in nature. Edelman himself notes that the assumption of separate and independent votes does not apply to the Modified District Plan for Electoral College in which a voter casts a single vote that counts in two ways, though he speculates that voting power under this plan may be just about the same as when votes are separate and independent if the number of voters is large enough. In any event, even if the Modified District Plan or the National Bonus were modified to *allow* separate and independent votes, voters would never have reason to *use* their new-found "flexibility" to "split" these votes, given the binary nature of Presidential election outcomes — that is to say, there is no reason to vote for a Democratic-pledged elector at the district (or state) level and a Republican pledged-elector at the state (or national) level (or vice versa).

This gives us some insight into why the Banzhaf-style calculations for the Modified District and National Bonus Plans allowed mean individual voting power to exceed what it would be under direct popular vote — they in effect assume, not only that voters *can* "split" their district (or state) and state (or national) votes in this manner, but also that they *actually do* "split" their votes *half the time*, thereby removing the correlation that would otherwise exist between district (or state) and state (or national) votes.

2. A Simple Example

As a warm-up exercise, let us consider the simplest case in which nine voters are partitioned into three uniform districts. Elections are held under four distinct voting rules, each of which is symmetric with respect to both voters and two candidates A and B. Under all rules, voters cast a single vote that counts in two ways, i.e., first in the 'district' tier and second in the 'at-large' tier. With the U.S. Electoral College in mind, we may refer to first-tier votes as 'popular votes' and second-tier votes as 'electoral votes.' These are the four voting rules:

(1) *Pure District System*: there is l electoral vote for each district, and the candidate winning a majority of electoral votes (2 out of 3) is elected;

(2) *Small At-Large Bonus System*: there is 1 electoral vote for each district plus 1 at-large electoral vote, and the candidate winning a majority electoral votes (3 out of 4) is elected (ties may occur in the second tier);

(3) *Large At-Large Bonus System*: there is 1 electoral vote for each district plus a bloc of 2 at-large electoral votes, and the candidate winning a majority of electoral votes (3 out 5) is elected; and

(4) *Pure At-Large System*: there no districts or, in any case, there is a bloc of 4 or more at-large electoral votes, so the districts are superfluous and the candidate winning a majority of the popular votes (e.g., 5 out of 9) is elected.

We consider things from the point of view of a focal voter *i* in District 1, who confronts $2^8 = 256$ distinct combinations of votes that may be generated by the other eight voters. We want to determine, for each voting rule, in how many of the 256 combinations voter *i* is *decisive*, in the sense that *i*'s vote tips the election outcome for one way or the other.[2] The number of such combinations is voter *i*'s *Banzhaf score*, and the number of such combinations divided by 256 is voter *i*'s (absolute) *Banzhaf voting power* in the two-tier voting game. If each combination is equally likely, voter *i*'s Banzhaf power is equal to the *probability* that *i* casts a vote that is *doubly decisive*, i.e., the individual vote is decisive in *i*'s district and/or at-large *and* the district vote and/or the at-large bloc is decisive in the second tier.

Table 1 lists all 256 possible district vote profiles, shows the number of distinct voting voter combinations giving rise to each profile, indicates for each whether voter *i*'s vote is decisive under each of the four rules, and reports voter *i*'s Banzhaf score and voting power for each rule. We see that Banzhaf voting power increases as the weight of the at-large component increases.[3] The bottom

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[2] If the number of voters *n* is even (e.g., $n = 100$), the *interpretation* of a decisive vote differs somewhat according to whether the voting context is parliamentary or electoral. Under usual parliamentary rules, a tie vote defeats a motion, so voter *i* decisive in any voting combination in which 50 other voters vote 'yes' and 49 vote 'no,' as the motion passes or fails depending on whether *i* votes 'yes' or 'no.' However, in elections between two candidates (our present concern), voting rules are typically neutral between the candidates, so a tie outcome might be decided by the flip of a coin. In this event, a voter *i* is "half decisive" in any voting combination in which 50 other voters vote for A and 49 for B (A wins if *i* votes for A and each candidate wins with .5 probability if *i* votes for B) *and also* in any voting combination in which 49 other voters vote for A and 50 for B. The upshot is that voter *i*'s total Banzhaf score (and voting power) is the same under either interpretation. Thus we can (and will) speak loosely "the probability of a tie vote" even when the number of voters is even. More obviously, we can (and will) speak interchangeably between "the probability of voter *i* breaking what would otherwise be a tie vote" and "the probability of a tie vote" when the number of voters is large.

[3] However, with only nine voters, the Large At-Large System is effectively equivalent to the Pure At-Large System, because the candidate who wins the at-large vote must win at least one district and thus 3 out of 5 electoral votes.

of the table shows Banzhaf voting power calculated in the manner of Edelman (2004), on the assumption that voters cast *separate and independent votes* at the district and at-large levels. In the Edelman setup, individual voting power is maximized with a mixture of district and at-large electoral votes such that the at-large component is approximately the square root of the total number of electoral votes. The Edelman setup does not generate an ordinary simple voting game, and therefore Edelman voting power values cannot be calculated in the manner of Table 1; however, they can be readily calculated, as shown in a footnote below the table. Any district vote profile may occur with any popular vote split and, in particular, a candidate can win the at-large vote without carrying any district.

Table 2 is derived from Table 1 and has two types of entries in each cell. First, it crosstabulates the 256 voting combinations with respect to whether the vote in *i*'s district (DV) is tied, thereby making *i*'s vote decisive within the district (column variable), and whether the at-large vote (ALV) is tied, thereby making *i*'s vote decisive with respect to the at-large vote (row variable). We call each cell a *contingency*; the lower number in each cell indicates number of voter combinations giving rise to that contingency. The contingencies themselves pertain to characteristics of the first-tier vote only. However, the four top numbers in each cell pertain to the four distinct second-tier voting rules and indicate, for each voting rule, the number of combinations in which *i*'s vote is doubly decisive, and thereby contributes to *i*'s Banzhaf score.

The numbers in Table 2 were determined by consulting Table 1, and Table 1 in turn was easy (if tedious) to construct. But if the number of voters expands even slightly, it becomes impractical to replicate Table 1 (for example, with 25 voters the number of possible combinations facing voter i is $2^{24} = 16,777,216$), so some less direct method for enumerating (or estimating) Banzhaf scores and voting power values must be devised. We now turn to a large-scale, though still simplified (relative to either Electoral College variant) example.

3. A Large-Scale Example with Uniform Districts

Let us now consider an example in which n =100,035 voters are uniformly partitioned into k = 45 districts with 2223 voters, each with a single electoral vote and with a bloc of 6 additional electoral votes elected at-large.[4]

We note two relevant baselines. Given 51 districts and no at-large seats and using the standard approximation $\sqrt{2 / \pi n}$ with n = 100,035/51 = 1961.47 for the probability of a tie vote, individual voting power within a district is .0180156. Using *Voting Power Algorithms*, the voting power of each district in the second tier is .112275. Thus the individual voting power (the probability of double decisiveness) is .0180156 × .112275 = .0020227. At the other extreme, with 25 or fewer districts (i.e., effectively direct popular vote), individual voting power is $\sqrt{2 / \pi n}$ with n = 100,035 or .0025227.

We begin with Table 3A, set up in the same manner as Table 2 and initially pertaining to first-tier votes only. Since the number of voting combinations is impossibly large, *proportions* rather than counts of combinations associated with each contingency are displayed and, given random voting, these are also the *probabilities* of each contingency.

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4 In Edelman's setup, individual voting power would be maximized with 44 districts plus 7 at-large votes, since 7 is the integer closest to the square root of the total number of 51 electoral votes.

We first calculate the probability that the popular vote is tied, which gives us the total in the first row. As noted just above, this probability is .0025227. Using the same approximation with n = 100,035/45 = 2223, we calculate the probability that the vote in *i*'s district is tied to be .0169227, which gives us the first column total. Subtraction from 1.0000000 gives us the totals in the second row and second column. These marginal proportions (or probabilities) are shown in Tables 3A-C.

So far as Edelman-style calculations are concerned, we are almost done. If district and at-large votes are separate and independent, we can calculate the probabilities of contingencies simply by multiplying the corresponding row and column probabilities, as shown in Table 3B. But, given Edelman's assumptions, we need not be concerned with the interior cells at all. We need look only at the marginal proportions in the first row and first column and then take account of voting at the second tier. Second-tier voting is given by the voting rule 46:51(26:6,1, . . . ,1) — that is, a weighted voting game with 46 players (45 districts plus the at-large bloc), a total of 51 electoral votes, a quota of 26 (a bare majority of the total of 51 electoral votes), and voting weights of 6 for the at-large bloc and 1 for each district. *Voting Power Algorithms* produces .628702 and .080083 as the voting power for the at-large bloc and each district respectively. The voting power of voter *i* through district representation is his probability of being decisive within his district times the probability that is district is decisive in the second tier, i.e., .0169227 × .080083 = .0013552, and *i*'s voting power through at-large representation is his probability of being decisive in the popular vote times the probability that the at-large bloc is district is decisive in the second tier, i.e., .00252227 × .628702 = .0015860. Within Edelman's setup, the overall voting power of each voter is simply the sum of these probabilities, i.e., .0029412, as shown in Table 3C.[5] Note that this is greater than voting power under direct popular election, i.e., .0025227. Figure 3 shows Edelman-style voting power for all magnitudes of at-large representation.

If, in contrast to the Edelman setup, each voter has a single vote that counts the same way for both district and at-large representation, we have an ordinary simple voting game, and individual two-tier voting power cannot exceed the .0025227 level resulting from direct popular (Pure At-Large) election (Felsenthal and Machover, 1998, pp. 58-59). However, voting power calculations become far more complex.

We first return to Table 3B and observe that, in the single-vote setup, the marginal probabilities are the same, as is shown in Table 4A. However, the fact that voters cast the same vote for both district and at-large representation induces a degree of correlation between the vote in any district and the at-large vote, so the probability that both votes are tied is greater than the .0000427 in the Edelman setup.

We can directly calculate the *conditional probability* that the at-large vote is tied given that a district vote is tied. Given that the vote in *i*'s district is tied, the overall at-large vote is tied if and only if there is also a tie in the residual at-large vote after the votes cast in voter *i*'s district are removed. The probability of this event is given by the standard approximation $\sqrt{2/\pi n}$, where n is now 100,035 − 2223 = 97812, and is equal to .0025512 as shown in Table 4A. We can now derive the unconditional probability that both types of ties occur simultaneously by multiplying this

5 Taking the sum of the voting powers associated with each of the voter's (district and at-large) votes may appear to double-count those voting combinations in Contingency 1 in which *both* of *i*'s two votes are doubly decisive, but at the same time it misses voting combinations in Contingency 1 in which *neither* vote by itself is doubly decisive but the two votes together are, and it turns out that these combinations exactly balance out (Beisbart, 2007).

conditional probability by the probability that the district vote is tied in the first place, i.e., .0025512 × .0167366 = .0000432. With this piece of the puzzle in place, the probabilities of the other contingencies are determined by subtraction, as shown in Table 4B. Comparing Tables 4B and 3B, we observe that the probabilities of the contingencies differ only slightly, with the probability of ties at one level but not the other being slightly less in the single-vote setup, so the substantially lower overall voting power arising from this setup relative to Edelman's must result mostly from the workings of second-tier voting.

In any event, voter *i* is decisive in the two-tier voting process only if the at-large and district votes are both tied (Contingency 1), the at-large vote only is tied (Contingency 2), or the district vote only is tied (Contingency 3). Having determined the probabilities of these contingencies, our next — and much more difficult — task is to determine, given each of these contingencies, the probability that voter *i*'s vote is decisive in the second tier as well.

First, let's form some general expectations. Contingency 1, being the conjunction of two already unlikely circumstances, is extraordinarily unlikely occur but, if it does occur, voter *i* is very likely to be doubly decisive. Voter *i* is doubly decisive if and only if neither candidate has won a majority of 26 electoral votes from the 44 other districts — put otherwise, if each candidate has won between 19 and 25 districts. By breaking a tie in both his district and at-large vote, voter *i* is tipping 7 electoral votes one way or the other and thereby gives one or other candidate the 26 electoral votes required for election. Given random individual voting, the electoral votes of the other 44 districts are likely to be quite evenly divided. Since each candidate is likely to have won about half of them, it likely that neither has won 26 out of 44 districts, thereby making voter *i* doubly decisive.

At this point, it may be tempting to note that the second tier voting game is 45:51(26:7,1,...,1) and that the Banzhaf voting power of the at-large plus voter *i*'s district bloc of 7 votes is .708785, and from this to conclude that voter *i*'s probability of double decisiveness through Contingency 1 is therefore .0000432 × .708785 = .0000306. However, to interpret the Banzhaf voting power of the at-large plus one district bloc calculated in this manner as the probability of second-tier decisiveness is to assume that all second-tier voting combinations are equally likely and, in particular, are independent of votes at the district level (as they are in the Edelman setup). But, given that the district and at-large votes consist of the same votes counting both ways, this assumption is not justified.

Contingency 2 is much more likely to occur than Contingency 1, while voter *i*'s probability of double decisiveness is only slightly less, since the voter is tipping almost as many electoral votes (6 rather than 7) one way in Contingency 2 as in Contingency 1. Voter *i* is now doubly decisive if and only if neither candidate has won a majority of 26 electoral votes from all 45 districts — put otherwise, if each candidate has won between 20 and 25 districts. By breaking an at-large vote tie, voter *i* is tipping 6 electoral votes one way or the other and thereby gives one or other candidate the 26 electoral votes required for election. Again, given random voting, the electoral votes of the 45 districts are likely to be quite evenly divided, so it quite likely that neither candidate has won 26 districts.

At this point, it may again be tempting to note that the second tier voting game is 46:51(26:6,1,...,1) and that the Banzhaf voting power of the at-large bloc is .628702, and from this to conclude that voter i's probability of double decisiveness through Contingency B is therefore $.0024800 \times .628702 = .0015592$. But again this assumes that all second-tier voting combinations are equally likely, and again this assumption is not justified.

Contingency 3 is still more likely to occur than Contingency 2, but voter i is far less likely to be doubly decisive in this contingency, since he is tipping only a single electoral vote one way or the other. Voter i is doubly decisive if and only if neither candidate has won a majority of 26 electoral votes from the other 44 districts *and* the at-large bloc of 6 votes, i.e., in the event that there is a overall 25-25 electoral vote tie. Such a tie results if and only if one candidate has carried 25 districts, while the other candidate has carried 19 districts *and* the at-large vote. The probability of such an event is very small for three reasons:

(1) an *exact* tie in the second-tier electoral vote tie is required, because i is tipping only a single electoral vote;

(2) the split in *district* electoral votes must be unequal in a degree that depends on the number of at-large seats (here 25 to 19 with 6 at-large seats) in order to create a tie in overall electoral votes, and such an unequal split is less likely than an equal split, since random voting always produce 50-50 expectations; and

(3) this rather unlikely 25-19 split in favor of one candidate in terms of district electoral votes must come about in the face of a popular vote majority in favor of the *other* candidate.

The last point implies that, in Contingency 3, voter i is doubly decisive only if i's vote can bring about the kind of *election inversion* (or 'reversal of winners,' 'wrong winner,' 'election conflict,' 'referendum paradox,' etc.) in which the candidate who wins with respect to district (but excluding at-large) electoral votes at the same time loses with respect to the overall at-large (popular) vote (Miller, 2010). It is characteristic of districted election systems such as U.S. Presidential elections and U.K. general elections that such election reversals may occur, but they are quite unlikely unless the (at-large or popular vote) election is very close. But we must bear in mind that almost all large-scale Bernoulli elections are extremely close. Indeed, if district and at-large votes are cast separately and independently in the Edelman manner so there is no correlation between them, it is evident that 50% of all random elections produce election reversals. This is shown in Figure 4A, which is based on a sample of 30,000 Bernoulli elections in which the at-large vote and the district votes were generated independently. In contrast, when the popular vote is the district vote summed over all districts, a substantial correlation is induced between district and at-large votes, which considerably reduces the incidence of election inversions. This is shown in Figure 4B, which is based on the same sample of 30,000 Bernoulli elections when the at-large popular vote is the sum of the district votes. In this sample, election inversions occurred in 20.4% of the elections, very closely matching the rate of 20.5% found by Feix et al. (2004) in a sample of one million Bernoulli (or 'Impartial Culture') elections. The correlation between the number of uniform districts carried by a candidate and the candidate's national popular vote is about +0.784. This degree of associations appears to be essentially constant regardless of the number of voters or districts, provided the latter is more than about 20 and the former is more than a thousand or so per district.

4. Bernoulli Election Simulations

Having formed expectations about the probability of double decisiveness in each contingency, we must now assign numbers to these probabilities. While it may be possible to proceed analytically, I have found the obstacles to be formidable and have instead proceeded on the basis of large-scale simulations. For the present case with 45 districts and 6 at-large seats, I have generated a sample of 1.2 million Bernoulli elections.[6]

The next question is how to use the results of the simulation to estimate the relevant probabilities. The most direct approach is to produce the crosstabulation depicted in Table 4C, which shows the absolute frequencies produced by these simulations. The number in the lower part of each cell is the number of times that the contingency arose. The number in the upper part of each cell is the number of times voter *i*'s was doubly decisive in that contingency. Overall, voter *i* was doubly decisive (DD) in 2970 elections out of 1,200,000. Thus the estimated *a priori* voting power of voter *i* (and every other voter as well, given the overall symmetry) is 2970/1,200,000 or .002475, a figure that sits comfortably between the lower bound of .0020227 for district only voting and the upper bound of .0025227 for direct popular voting noted at the outset of this section. Our confidence in this estimate is reinforced by comparing Table 4D, in which all absolute frequencies in Table 4C are converted into proportions (and estimated probabilities), with Table 4B. It is evident that the relative frequency of each contingency closely matches the exact probabilities calculated earlier.

A second approach is to replace the estimated probabilities of each contingency in the lower part of each cell in Table 4D with the known probabilities displayed in Table 4B. In this case, the numbers are so similar that the substituion makes essentially no difference, as voter *i*'s estimated voting power becomes .002488, in contrast to .002475 using simulated data only.

A third approach is suggested if we examine the frequency distributions underlying Table 4C. Figure 5A shows the frequency distribution of districts won by Candidate A in the 51 elections in which both the district and at-large votes are tied. Voter *i* is doubly decisive provided that the number of districts won by either candidate lies with in the range of 19-25. This was true in 49 elections out of the 51 elections, giving voter *i* a .960785 probability of double decisiveness in this contingency. But it evident that another sample of 1.2 million Bernoulli elections (including about 50 belonging to Contingency 1) would likely produce a rather different statistic. And, given a large enough sample size, we would expect this distribution to fit a more or less normal pattern, rather than the bimodal pattern that happens to appear in the Figure 4A. So, given the present sample of elections, a more reliable estimate of voter *i*'s probability of double decisiveness may be derived by supposing that the underlying distribution of districts won by Candidate A is normally distributed with a known mean of 22 (i.e., one half of the 44 districts other than voter *i*'s), rather that the sample

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6 Each of the 45 districts has 2223 voters, a number selected so that both district and at-large vote ties may occur before focal voter *i* (in District 1) casts his vote and so that no ties occur after *i* has voted. The simulations, which are generated by SPSS syntax files, operate at the level of the district: the vote for candidate A in each district is a number drawn randomly from a normal distribution with a mean of 2223/2 = 1111.5 and a standard deviation of $\sqrt{.25 \times 2223}$ and then rounded to the nearest integer.

statistic of 22.294118, and with the standard deviation of 2.032674 found in this sample. (From this point of view, the main purpose of the simulation is to provide an estimate of this standard deviation.) The estimated proportion of times voter *i* is doubly decisive is therefore equal the proportion of the area under a normal curve that lies within 3.5/2.032674 = 1.72187 standard deviations from the mean, which is .914907. This suggests that the direct result of the simulation of .960785 is too high, and indeed Figure 4A suggest that it was only by 'good luck' that Candidate A never won fewer than 19 districts.

In like manner, Figure 5B shows the frequency distribution of districts won by Candidate A in the contingency that the at-large vote only is tied. The actual distribution closely matches a normal with a mean of 22.5 (i.e., one half of all the 45 districts). Given the much larger (2960) sample of elections in Contingency 2, it is unsurprising that the normal curve approach to estimating voter *i*'s double decisiveness produces almost the same result (.859592) as the sample statistic itself (.862838).

Figure 5C shows the frequency distribution of districts won by Candidate A in the contingency that the district vote only is tied. Since the distribution is clearly bimodal (which results from the fact that the at-large vote is not tied as in Charts 4 and 5 and one or other candidate has won the block of 6 at-large votes), we cannot use the normal curve approach. However, Contingency 3 is by far the most likely of the three contingencies that allow voter *i* to be doubly decisive, so the sample size is very large (n = 20,167) and the sample statistic for a 25-25 electoral vote tie (367/20,167 = .0181981) is itself quite reliable.

Putting this altogether in Table 4E, using the normal curve approach to estimate probabilities of double decisiveness in Contingencies 1 and 2 and the sample statistic in Contingency 3 and the known probabilities for the contingencies themselves, we get an estimate of voter i's voting power of .002465, compared with .002475 using sample statistics only (and .002488 using the sample statistics for probabilities of decisiveness in conjunction with the known probabilities for the contingencies themselves). In sum, we can be pretty confident that the true value of voter *i*'s voting power is about .00247, putting it slightly but clearly below the value of .002523 that results from direct popular vote. This contrasts of the Edelman value of .0029412 that results when voters cast separate and independent votes at the district and at-large levels.

Comparing Figures 5A-C with Figure 6A-C that result in the Edelman setup makes evident how the Edelman setup produces a greater probability of double decisiveness. We see that each contingency occurs with essentially the same probability in the two setups (as we saw before in the calculations displayed in Tables 3B and 4B). And in the first two contingencies, a voter is actually less likely to be doubly decisive in the Edelman setup, as the spread in districts won by either candidate is substantially larger. This results from correlation between popular votes won and number of districts won that results when each voter casts a single vote that counts twice (Figure 4B) rather that two separate and independent votes (Figure 4A). But this effect is more than wiped out in Contingency 3, where two setups result in quite different distributions of electoral votes won. In the single-vote setup, the distribution is strikingly bimodal (the distance between the modes depending on the number of at-large electoral votes relative to the total) because, as a candidate wins more districts, he is more likely to win the at-large vote as well, whereas in the Edelman setup no such correlation exists. Given the parameters we are working with (6 at-large electoral votes out of 51), the Edelman setup produces a distribution that is unimodal but, relative to a normal curve, slightly "squashed" in the center (Figure 6C). If the relative magnitude of the at-large component were increased, the "squashing" effect would be increased and would in due course produce bimodality, but it would always be substantially less than in the single-vote setup with the same at-large component. Thus, unless at-large component is wholly controlling (e.g., 26 electoral votes out of 51), the Edelman setup makes a even split of electoral votes far more likely than does the single-vote setup and thereby greatly enhances the probability of double decisiveness in Contingency 3, which in turn is by far the most probable contingency that (in either setup) allows double decisiveness.

I have duplicated the same kinds of simulations, but (for practical reasons) with smaller samples, for other odd values of the at-large component within a fixed total of 51 electoral votes. The results are displayed in Figure 7.[7] Simulations were run in blocks of 300,000 elections, and the voting power estimates are displayed for each sample (along with the means) for each at-large magnitude. It is evident that even blocks of this size produce considerable sampling error because few elections entail ties at either the district or national level, but the general pattern of the relationship between the magnitude of the at-large component and individual power is very clear and in sharp contrast with the pattern of the same relationship in the Edelman setup shown in Figure 3.

5. The National Bonus Plan for the U.S. Electoral College

The previous analysis pertained to a *uniform* district plus at-large voting system — that is, one which all the districts have the same number of voters and electoral votes. The most direct Electoral College application of the kind of analysis set out above pertains to variants of the National Bonus Plan, under which 538 electoral votes are cast in the present manner but the *national* popular vote winner is awarded a *bonus* of some number of electoral votes.[8] However, here the 'districts' (i.e., the states) are not uniform, having different number of voters and electoral voters.

Just as under the previous example, votes count in two distinct upper tiers (i.e., the voter's state and the nation as a whole), with the result that doubly decisive votes can arise in three distinct contingencies: (1) a vote is decisive at both the state and national levels and the combination of the state's electoral votes and the national bonus is decisive in the Electoral College; (2) a vote is decisive at the state level only and the state's electoral votes are decisive in the Electoral College; (3) a vote is decisive at the national level only and the national bonus is decisive in the Electoral College. However, under the bonus plan, the relevant probabilities and simulation estimates must

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7 The vertical axis in Figure 7 (and 9A and 9B) must show actual, rather than rescaled, voting power, because the voting power of the least favored voters varies as the at-large component (or national bonus) varies.

8 Though this idea had been around earlier, it was most notably proposed by Arthur Schlesinger, Jr. (2000) following the 2000 election. He proposed a national bonus of 102 electoral votes --- two for each state plus the District of Columbia. However, given an even number (538) of 'regular' electoral votes, it would seem sensible to make the bonus an odd number in order to definitively eliminate the possibility of electoral vote ties (though such ties would be far less likely given a large even-number bonus than with no bonus. It is clear that the motivating purpose for a national bonus is to reduce the probability of election inversions, not to redistribute voting power

be separately determined for voters in each state, each with its own number of voters and electoral votes. While the calculations and simulations are in this respect more burdensome, the procedure is a straightforward extension of that set out in the previous section. These simulation results were based on a sample of 256,000 Bernoulli elections. Sampling error presumably accounts for the relative minor anomalies in the following charts, but again the overall pattern is clear enough.

Figure 8A displays individual voting power, when calculated in the Banzhaf/Edelman manner, under a National Bonus Plan with a bonus of 101 electoral votes for the national popular vote winner. At first blush, Figure 8A may looks very similar to Figure 1 for the existing Electoral College. But inspection of the vertical axis reveals that the inequalities between voters in large and small states are considerably compressed relative to the existing system. Moreover, the same anomaly occurs here as with Banzhaf's calculations for the Modified District Plan, in that mean individual voting power (considerably) exceeds that under direct popular vote. Figure 8B displays individual voting power with a 101 electoral vote national bonus calculated in the manner set out in the last section. (It should be borne in mind that the plotted points in Figure 8B, like Figures 7, 9B and 10A, are estimates subject to some sampling error.)

Figure 9A displays individual voting power with a national bonus of varying magnitude, again calculated in the Banzhaf/Edelman manner, while Figure 9B shows the same when voting power is measured in the manner set out here. A bonus of zero is equivalent to the existing Electoral College system and a bonus of at least 533 (like an at-large component of four or more electoral votes in the example in Section 3) is logically equivalent to direct popular vote.[9] However, Figure 9B indicates that a bonus greater than about 150 is essentially equivalent to direct popular vote.

6. The Modified District Plan for the U.S. Electoral College

Under the Modified District Plan, a candidate wins one electoral vote for each Congressional District he carries and two electoral vote for each state he carries.[10] Individual voting power within each state is equal, because (we assume) each district has an equal number of voters. All districts have equal voting power in the Electoral College, because they have equal weight, i.e., 1 electoral vote; and all states have equal voting power in the Electoral College, because they have equal weight, i.e., 2 electoral votes. But individual voting power across states is not equal, because districts in different states have different numbers of voters (because House seats must be apportioned in whole numbers) and states with different numbers populations (and numbers of voters) have equal electoral votes.

The Modified District Plan is more complicated than it may at first appear. As in the

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9 With each vote counting the same way at the state and national levels, the national popular vote winner must win at least one state with at least 3 electoral votes, and 533 is the smallest number B such $B + 3 > .5 (538 + B)$.

10 This system is used at present by Maine (since 1972) and Nebraska (since 1992). The 2008 election for the first time produced a split electoral vote in one of these states, namely Nebraska, where Obama carried one Congressional District. (The Republican-dominated legislature may now switch state law back to winner-take-all.) A proposed constitutional amendment (the Mundt-Coudert Plan) in the 1950s would have mandated the Modified District Plan for all states.

previous discussions, doubly decisive votes can be cast in three distinct contingencies: (i) a vote is decisive in both the voter's district and state and the combined three electoral votes are decisive in the Electoral College; (ii) a vote is decisive in the voter's state and the state's two electoral votes are decisive in the Electoral College; and (iii) a vote is decisive in the voter's district and the district's one electoral vote is decisive in the Electoral College.

Moreover, because each individual vote counts in two ways, there are logical inter-dependencies in the way in which district and state electoral votes may be cast. Whichever candidate wins the two statewide electoral votes must also win at least one district electoral vote but, at the same time, need not win more than one. Thus, in a state with a single House seat, individual voting power under the Modified District Plan operates in just the same way as under the existing Electoral College, as its three electoral votes are always cast in a winner-take-all manner for the state popular vote winner. In a state with two House seats, the state popular vote winner is guaranteed a majority of the state's electoral votes (i.e., either 3 or 4) and a 2-2 split is precluded. In a state with three or more House seats, electoral votes may be split in any fashion and, in a state with five or more House seats, the statewide popular vote winner may win only a minority of the state's electoral votes — that is, 'election inversions' may occur at the state, as well as the national, level.

However, the preceding remarks pertain only to logical possibilities. Probabilistically, the casting of district and statewide electoral votes will to some degree be aligned in Bernoulli (and even more so in actual) elections. Given that a candidate wins a given district, the probability that the candidate also wins statewide is greater than 0.5 — that is to say, even though individual voters cast statistically independent votes, the fact that they are casting individual votes that count in the same way in two tiers (districts and states) induces a correlation between popular votes at the district and state levels within the same state. This correlation, which as we have seen is perfect in the states with only one House seat, diminishes as a state's number of House seats increases, and therefore enhances individual voting power in small states relative to what it is under the Pure District Plan.

Again we follow the procedure outlined earlier. In this case I generated a sample of 120,000 Bernoulli elections, with electoral votes awarded to the candidates on the basis of the Modified District Plan.[11] This generated a database that can be manipulated to determine frequency distributions of electoral votes for the focal candidate under specified contingencies with respect to first-tier voting, from which relevant second-tier probabilities can be inferred.[12]

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[11] The simulation took place at the level of the 436 districts, not individual voters. For each Bernoulli election, the popular vote for the focal candidate was generated in each Congressional District by drawing a random number from a normal distribution with a mean of $n/2$ and a standard deviation of $\sqrt{2/\pi n}$, i.e., the normal approximation of the symmetric Bernoulli distribution), where n is the number of voters in the district. (Of course, the other candidate won the residual vote.) The winner in each district was determined, the district votes in each state were added up to determine the state winner, and electoral votes are allocated accordingly.

[12] Even with the very large sample, few elections were tied at the district or state level, so the relevant electoral vote distributions were taken from a somewhat wider band of elections, namely those that fell within 0.2 standard deviations of an exact tie. (In a standard normal distribution, the ordinate at ±0.2×SDs from the mean is about .98 times that at the mean.)

Figure 10A shows individual voting power across the states under the Modified District Plan. This chart invites comparison with Figure 10B, which depicts individual voting by state population under the Pure District Plan. It can be seen that the 'bloc effect' for the smallest states with three electoral votes, and the 'semi-bloc effect' for the next smallest states, under the Modified District Plan enhances the voting power of voters in these small states relative to that under the Pure District Plan. Figure 10A also invites comparison with Figure 2, which depicts individual voting power by state population under the Modified District plan when calculated in the Banzhaf/Edelman manner. Inequality in voting power is slightly less in Figure 10A. However the main difference is that the (absolute and not rescaled) voting power of all voters in substantially less in Figure 10A, as is indicated by the position of the line showing (rescaled) individual voting power under direct popular vote.

All Possible Vote Profiles (Bipartitions) Confronting a Focal Voter *i* in District 1, Given a Total of Nine Voters Uniformly Partitioned into Three Districts

			Number of Times Voter i is Decisive (Total is i's Bz Score)			
Pop. Vote	***District Vote Profile***	***n****	DV	DV + 1 AL	DV + 2 AL	PV (all AL)
8-0	(2-0) (3-0) (3-0)	1	0	0	0	0
	Total	**1**	**0**	**0**	**0**	**0**
7-1	(1-1) (3-0) (3-0)	2	0	0	0	0
	(2-0) (2-1) (3-0)	3	0	0	0	0
	(2-0) (3-0) (2-1)	3	0	0	0	0
	Total	**8**	**0**	**0**	**0**	**0**
6-2	(0-2) (3-0) (3-0)	1	0	0	0	0
	(1-1) (2-1) (3-0)	6	0	0	0	0
	(1-1) (3-0) (2-1)	6	0	0	0	0
	(2-0) (3-0) (1-2)	3	0	0	0	0
	(2-0) (2-1) (2-1)	9	0	0	0	0
	(2-0) (1-2) (3-0)	3	0	0	0	0
	Total	**28**	**0**	**0**	**0**	**0**
5-3	(0-2) (3-0) (2-1)	3	0	0	0	0
	(0-2) (2-1) (3-0)	3	0	0	0	0
	(1-1) (3-0) (1-2)	6	6	3**	0	0
	(1-1) (2-1) (2-1)	18	0	0	0	0
	(1-1) (1-2) (3-0)	6	6	3**	0	0
	(2-0) (3-0) (0-3)	1	0	0	0	0
	(2-0) (2-1) (1-2)	9	0	0	0	0
	(2-0) (1-2) (2-1)	9	0	0	0	0
	(2-0) (0-3) (3-0)	1	0	0	0	0
	Total	**56**	**0**	**6****	**0**	**0**

4-4	(0-2) (3-0) (1-2)	3	0	1.5**	3	3
	(0-2) (2-1) (1-2)	9	0	4.5**	9	9
	(0-2) (1-2) (3-0)	3	0	1.5**	3	3
	(1-1) (3-0) (0-3)	2	2	2	2	2
	(1-1) (2-1) (1-2)	18	18	18	18	18
	(1-1) (1-2) (2-1)	18	18	18	18	18
	(1-1) (0-3) (3-0)	2	2	2	2	2
	(2-0) (2-1) (0-3)	3	0	1.5**	3	3
	(2-0) (1-2) (1-2)	9	0	4.5**	9	9
	(2-0) (0-3) (2-1)	3	0	1.5*	3	3
	Total	**70**	**40**	**55**	**70**	**70**
3-5	**Dual of 5-3**	**56**	**12**	**6**	**0**	**0**
2-6	**Dual of 6-2**	**28**	**0**	**0**	**0**	**0**
1-7	**Dual of 7-1**	**8**	**0**	**0**	**0**	**0**
0-8	**Dual of 8-0**	**1**	**0**	**0**	**0**	**0**
	Total [Bz Score]	**256**	**64**	**67**	**70**	**70**
	Bz Power		**.25**	**.26172**	**.27344**	**.27344**
	Edelman Bz Power* **		**.25**	**.29004**	**.33008**	**.27244**

* *n* is the number of distinct voter combinations giving rise to the specified district vote profile.

** In these profiles, Banzhaf awards voter *i* "half credit" as *i*'s vote is decisive with respect to whether a particular candidate wins or there is a tie between the two candidates. (Under the other voting rules, ties cannot occur.)

*** Edelman Bz Power = Prob. i decisive in district × Prob. district decisive in Tier 2 + Prob. i decisive at-large × Prob at-large decisive in Tier 2

AL = 1: .5 × .375 + .27344 × .375 = .29004

AL = 2: .5 × .25 + .27344 × .7 = .33008

TABLE 1

	DV Tied	DV Not Tied	Total
ALV Tied	40 / 40 / 40 / 40 **Contingency 1** 40	0 / 15 / 30 / 30 **Contingency 2** 30	40 / 55 / 70 / 70 70
ALV Not Tied	24 / 12 / 0 / 0 **Contingency 3** 88	0 / 0 / 0 / 0 **Contingency 4** 98	24 / 12 / 0 / 0 186
Total	64 / 52 / 40 / 40 128	0 / 15 / 30 / 30 128	64 / 67 / 70 / 70 256

All District / 1 A-L / 2 A-L / All A-L

TABLE 2

	DV Tied	DV Not Tied	Total
ALV Tied	**Contingency 1**	**Contingency 2**	.0025227
ALV Not Tied	**Contingency 3**	**Contingency 4**	.9974773
Total	.0169227	.9830773	1.000000

TABLE 3A

	DV Tied	DV Not Tied	Total
ALV Tied	.0000427	.0024800	.0025227
ALV Not Tied	.0168800	.9805973	.9974773
Total	.0169227	.9830773	1.0000000

TABLE 3B

	DV Tied	DV Not Tied	Total
ALV Tied	.0000427	.0024800	× .628702 = .0015860 .0025227
ALV Not Tied	.0168800	.9805973	.9974773
Total	× .080083 = .0013552 .0169227	.9830773	.0029412 1.0000000

TABLE 3C

	DV Tied	DV Not Tied	Total
ALV Tied	.0025512 $\Downarrow$ .0000432		.0025227
ALV Not Tied			.9974773
Total	.0169227	.9830773	1.0000000

TABLE 4A

	DV Tied	DV Not Tied	Total
ALV Tied	.0000432	.0024795	.0025227
ALV Not Tied	.0168795	.9805978	.9974773
Total	.0169227	.9830773	1.0000000

TABLE 4B

	DV Tied	DV Not Tied	Total
ALV Tied	49 Prob. of DD = .960784 51	2554 Prob. of DD = .862838 2960	2603 3011
ALV Not Tied	367 Prob. of DD = .018198 20,167	0 Prob. of DD = .000000 1,176,822	367 1,196,989
Total	416 20,218	2554 1,179,782	2970 Prob. of DD = .002475 1,200,000

TABLE 4C

	DV Tied	DV Not Tied	Total
PV Tied	.0000408 .0000425	.0021283 .0024667	.0021692 .0025092
PV Not Tied	.0003058 .0168058	.0000000 .9806850	.0003058 .9974908
Total	.0003467 .0168483	.0021283 .9831517	.0024750 1.0000000

TABLE 4D

	DV Tied	DV Not Tied	Total
ALV Tied	× .914907 = .0000389 .0000425	× .859592 = .0021204 .0024667	.0021593 .0025092
ALV Not Tied	× .018198 = .0003058 .0168058	.0000000 .9806850	.0003058 .9974908
Total	.0003447 .0168483	.0021204 .9831517	.0024651 1.0000000

TABLE 4E

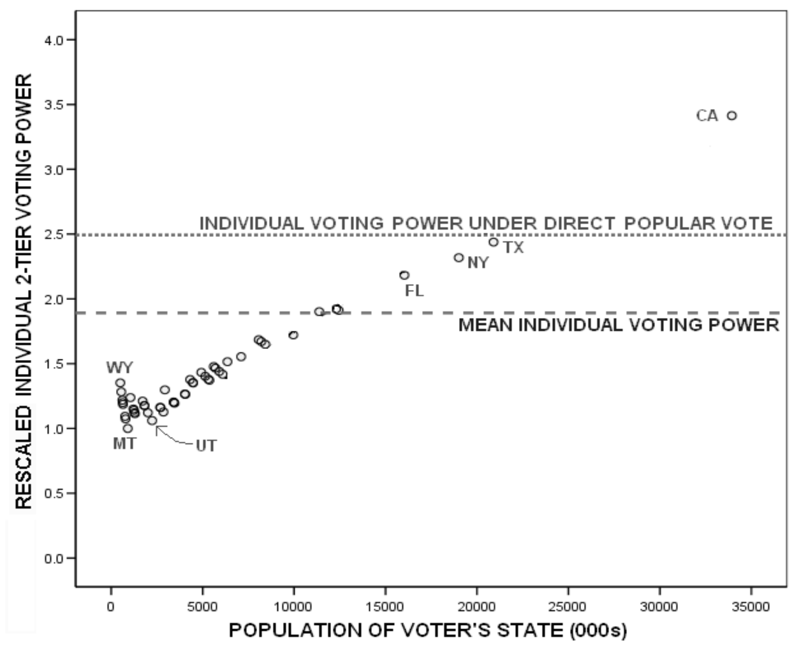


Figure 1 Individual Voting Power by State Population under the Existing Apportionment of Electoral Votes

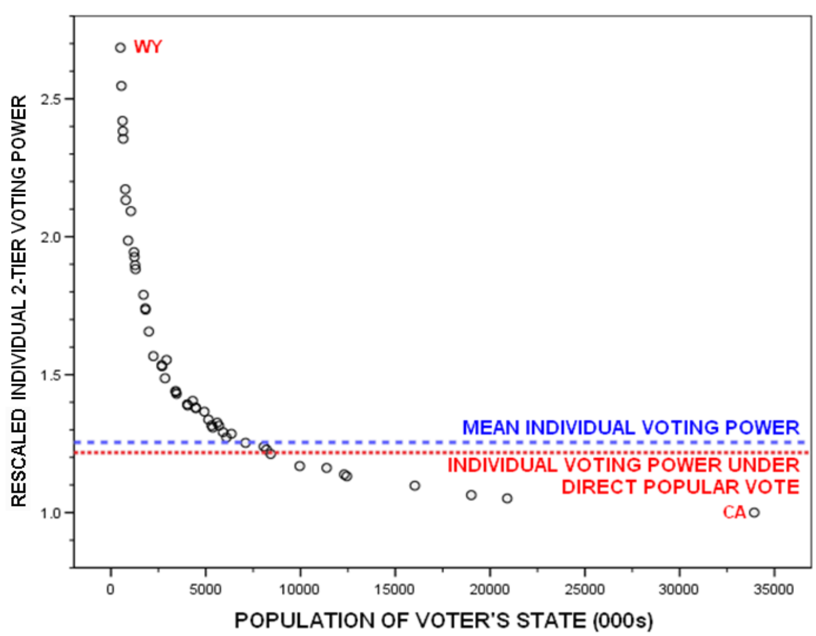


Figure 2 Individual Voting Power by State Population under the Modified District Plan (Banzhaf Calculations)

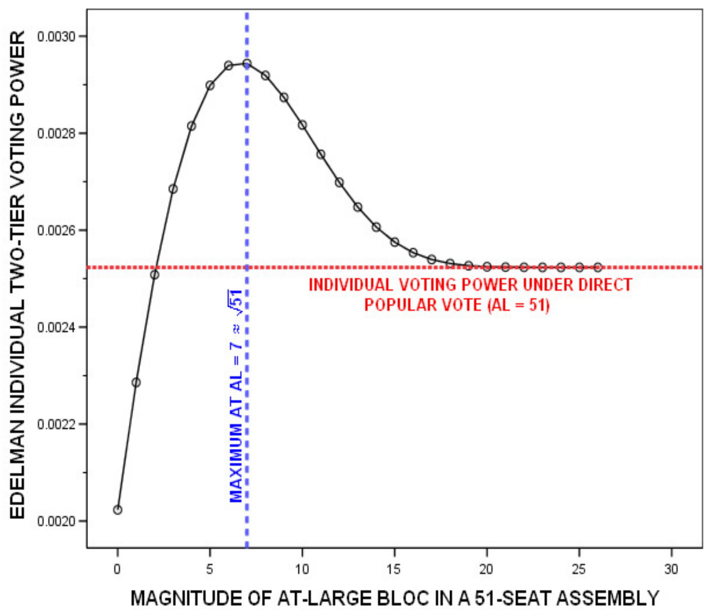


Figure 3 Individual Voting Power by Magnitude of of the At-Large Bloc (Edelman Calculations)

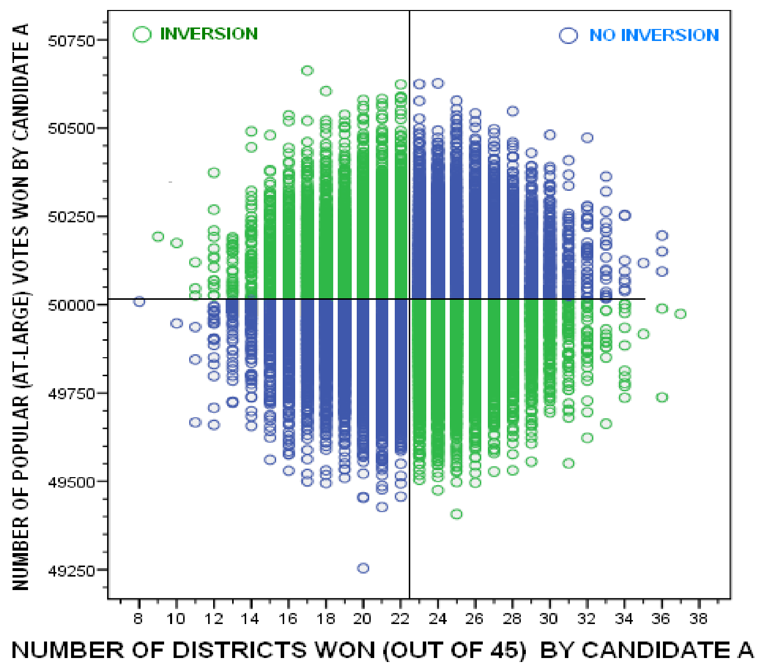


Figure 4A Two-Tier Bernoulli Election Outcomes with Separate and Independent Votes in Each Tier

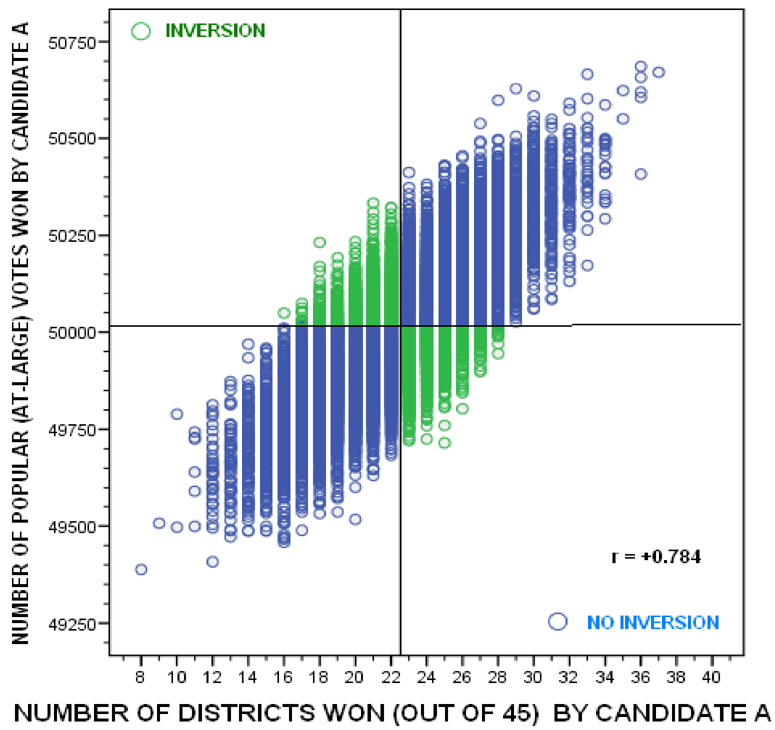


Figure 4B Two-Tier Bernoulli Election Outcome when One Vote Counts the Same Way in Both Tiers

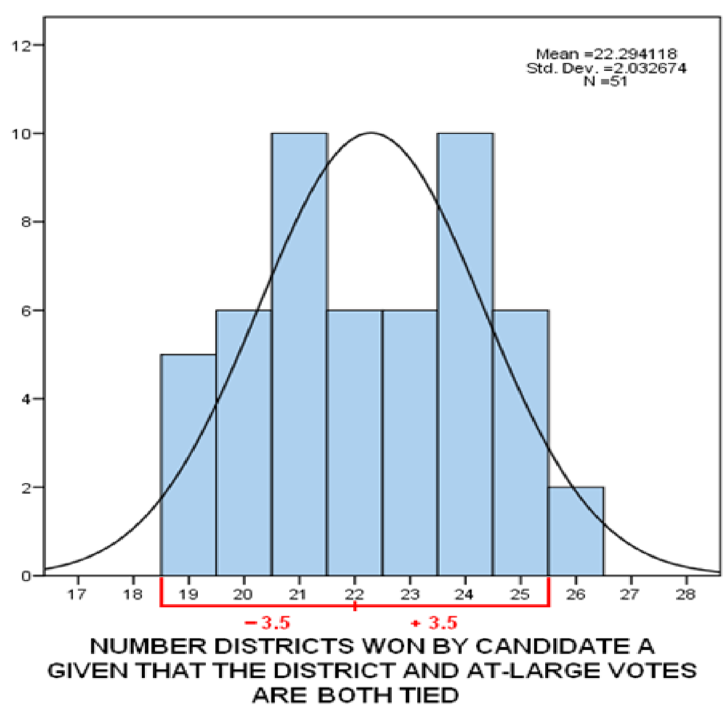
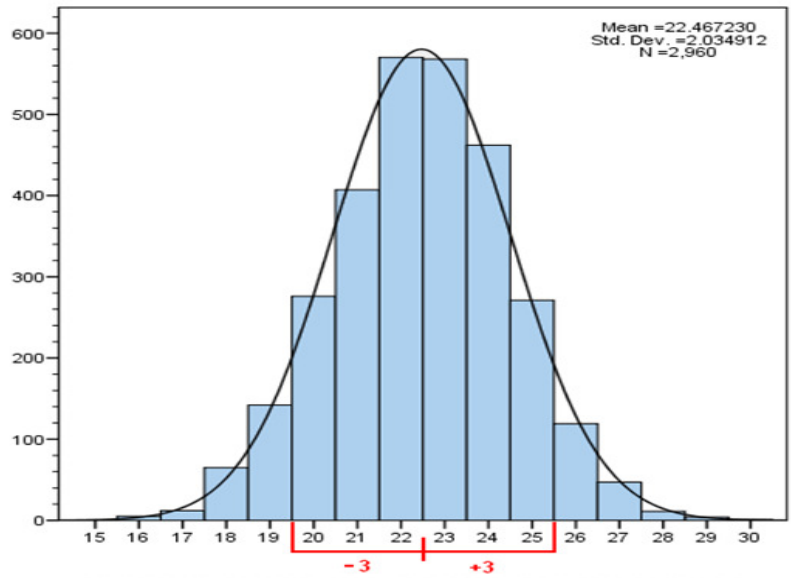
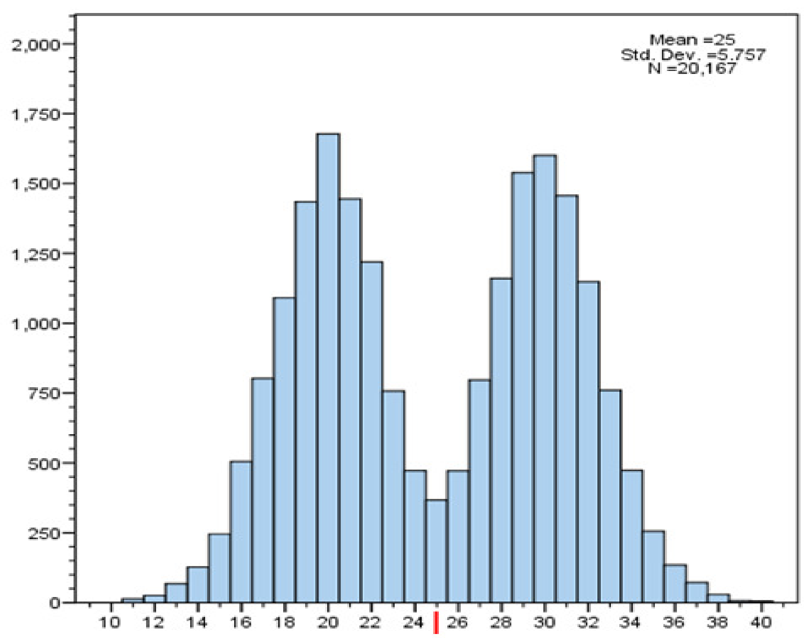


Figure 5A Distribution in Contingency 1



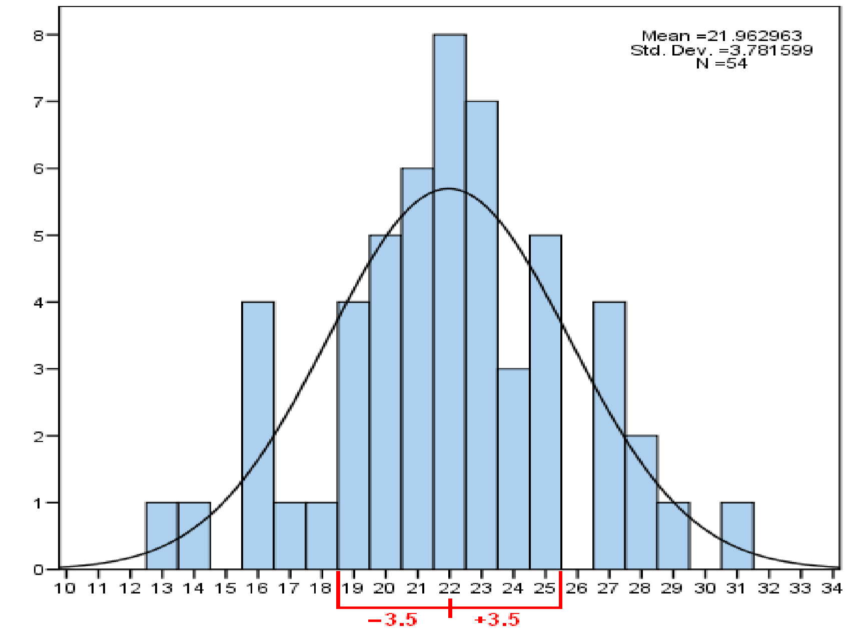
NUMBER OF DISTRICTS WON BY CANDIDATE A GIVEN THAT THE AT-LARGE VOTE ONLY IS TIED

Figure 5B Distribution in Contingency 2



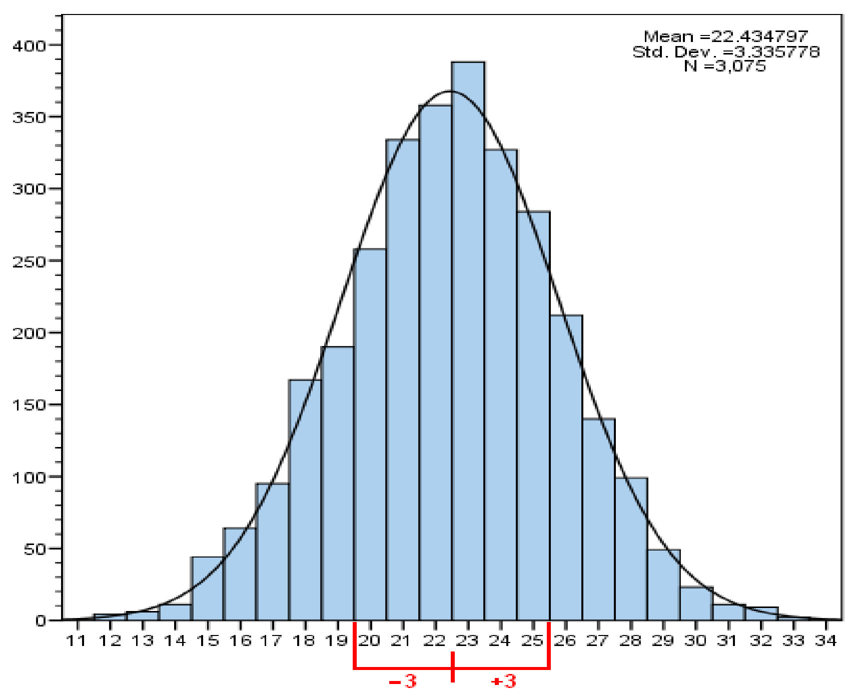
TOTAL NUMBER OF (DISTRICT + AT-LARGE) ELECTORAL VOTES WON BY CANDIDATE A GIVEN THE THE DISTRICT VOTE ONLY IS TIED

Figure 5C Distribution in Contingency 3



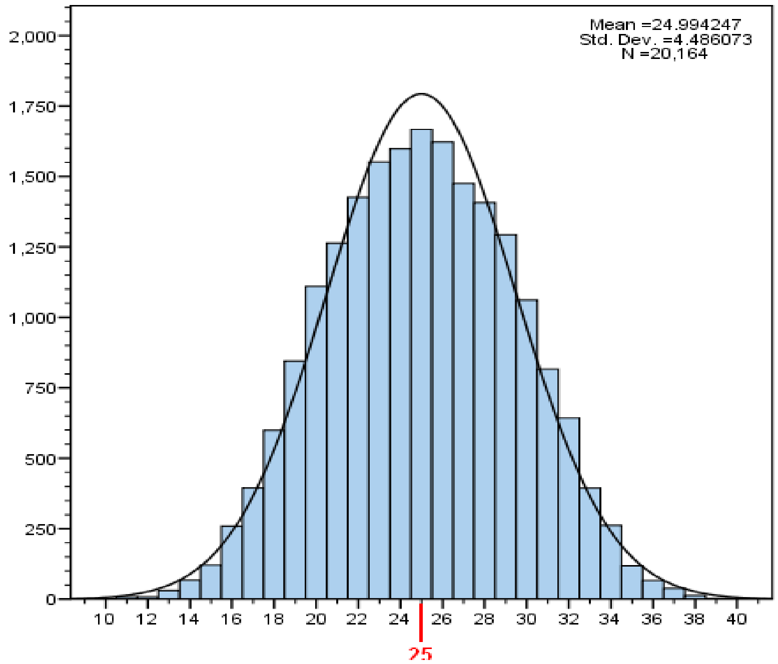
NUMBER DISTRICTS WON BY CANDIDATE A
GIVEN THAT THE DISTRICT AND AT-LARGE VOTES
ARE BOTH TIED (EDELMAN SETUP)

Figure 6A Distribution in Contingency 1



NUMBER OF DISTRICTS WON BY CANDIDATE A
GIVEN THAT THE AT-LARGE VOTE ONLY IS TIED
(EDELMAN SETUP)

Figure 6B Distribution for Contingency 2



TOTAL NUMBER OF (DISTRICT + AT-LARGE) ELECTORAL VOTES WON BY CANDIDATE A GIVEN THE THE DISTRICT VOTE ONLY IS TIED (EDELMAN SETUP)

Figure 6C Distribution in Contingency 3

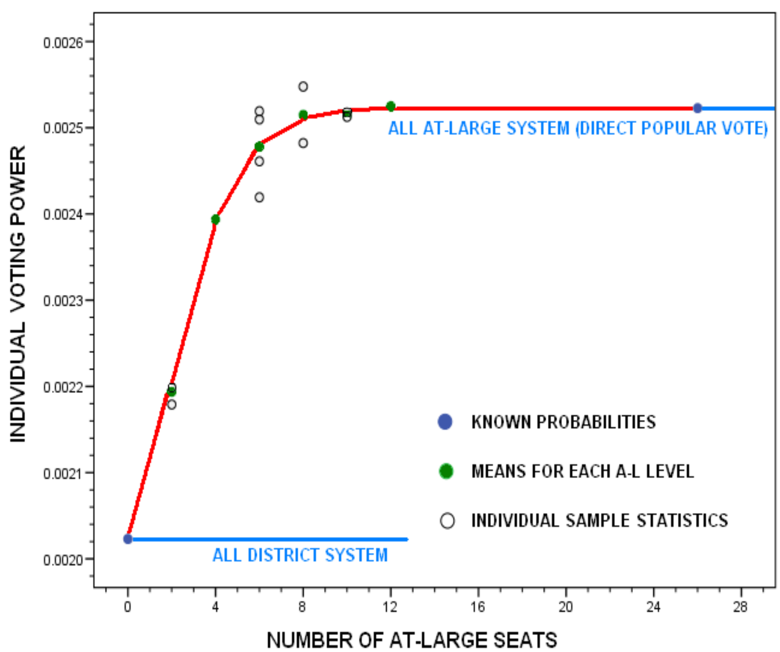


Figure 7 Indivial Voting Power By Magnitude of At-Large Component

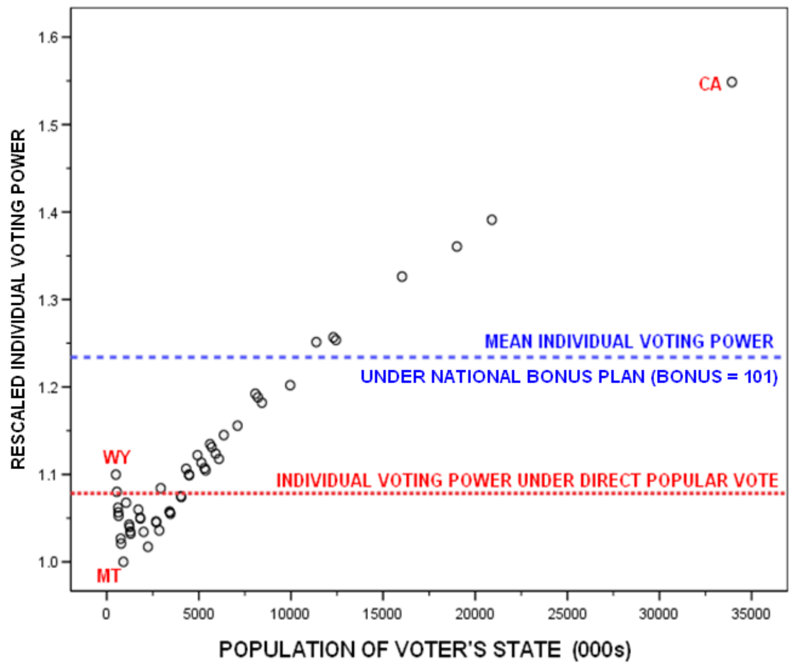


Figure 8A Individual Voting Power by State Population under the National Bonus Plan (Banzhaf/Edelman Calculations)

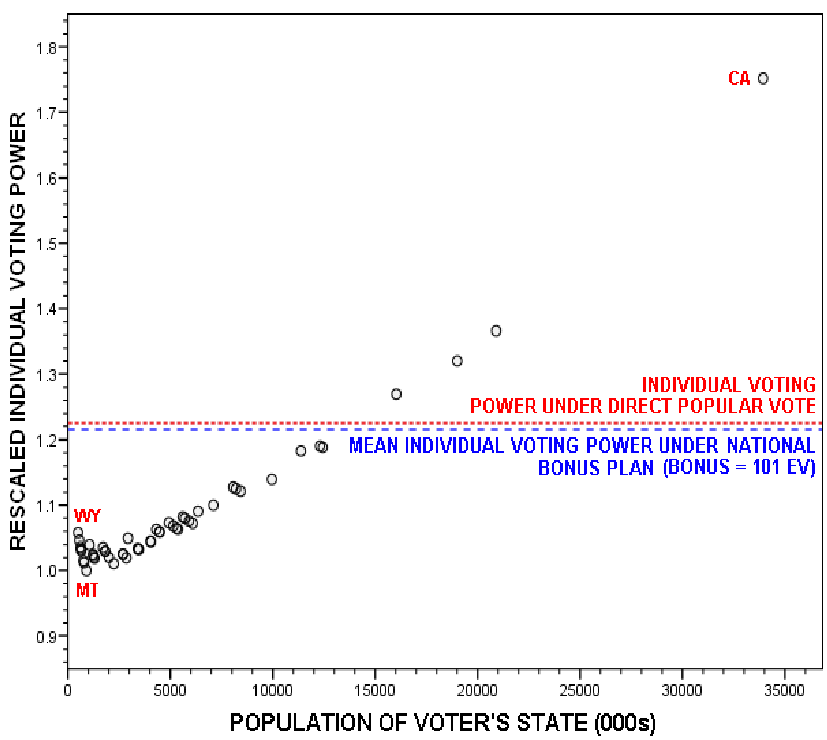


Figure 8B Individual Voting Power under the National Bonus Plan (Bonus = 101)

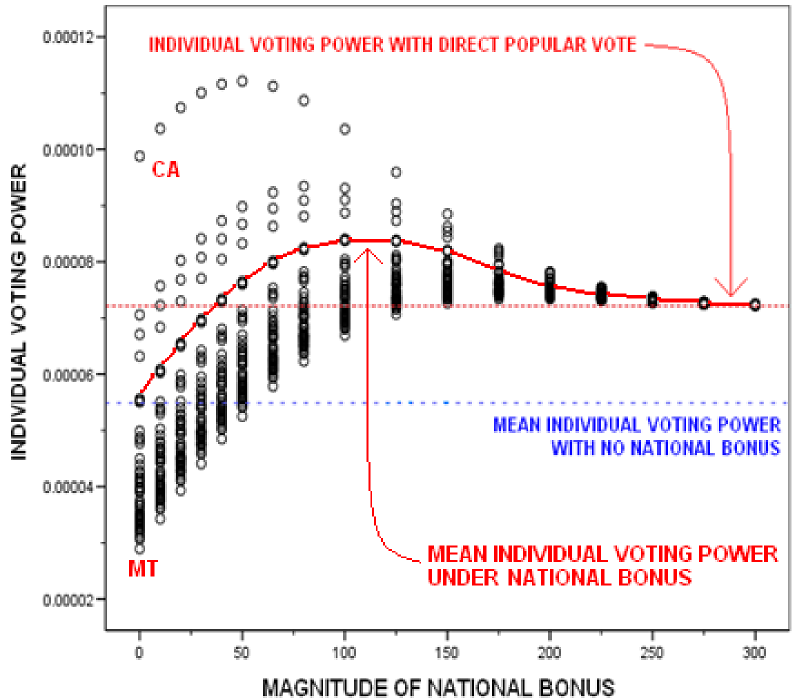


Figure 9A Individual Voting Power by Magnitude of National Bonus (Banzhaf/Edelman Calculations)

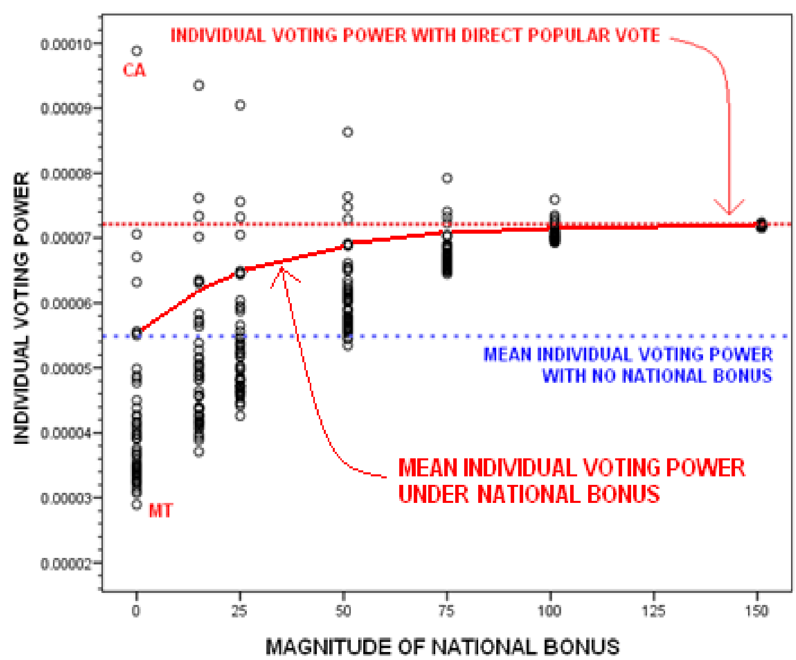


Figure 9B Individual Voting Power by Magnitude of National Bonus

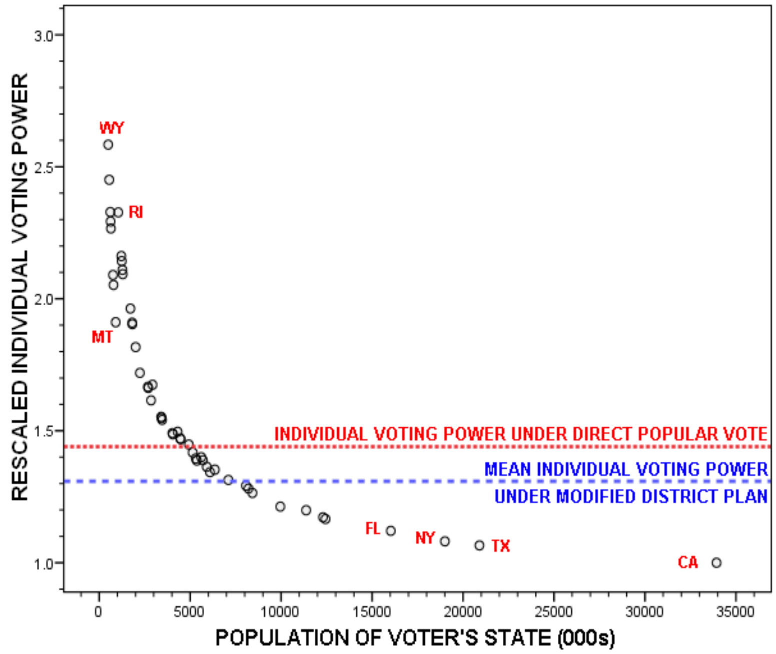


Figure 10A Individual Voting Power by State Population under the Modified District Plan

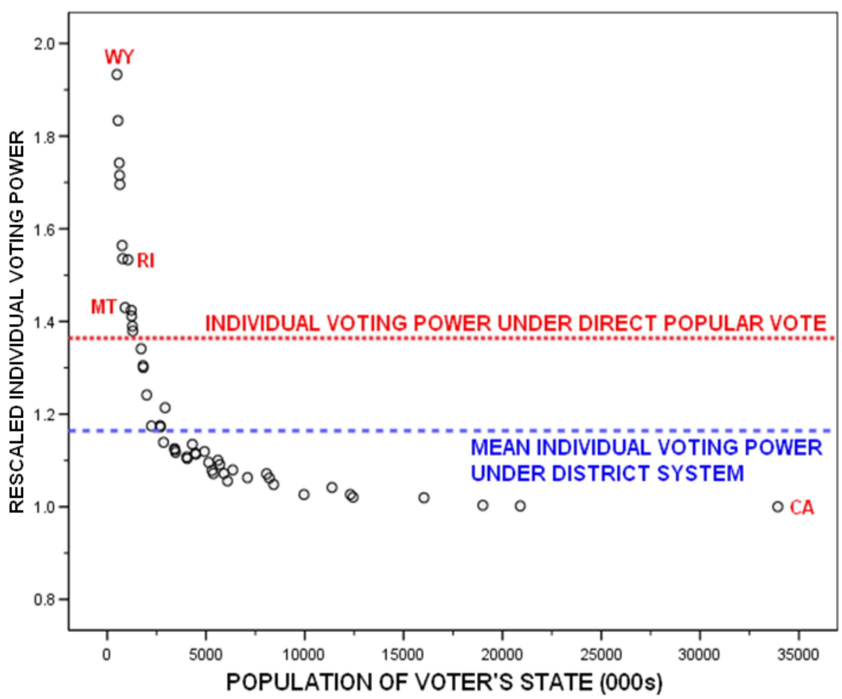


Figure 10B Individual Voting Power by State Population under the Pure District Plan

References

Banzhaf, John F., III (1968) "One Man, 3.312 Votes: A Mathematical Analysis of the Electoral College." *Villanova Law Review*, 13: 304-332.

Beisbart, Claus (2007). "One Man, Several Votes," University of Dortmund.

Beisbart, Claus, and Luc Bovens (2008). "A Power Measure Analysis of the Amendment 36 in Colorado." *Public Choice*, 134: 231-246.

Edelman, Paul H. (2004). "Voting Power and At-Large Representation," *Mathematical Social Sciences*, 47/2: 219-232.

Feix, Marc R., Dominique Lepelly, Vincent R. Merlin, and Jean-Louis Rouet (2004). "The Probability of Conflicts in a U.S. Presidential Type Election," *Economic Theory*, 23: 227-257.

Felsenthal, Dan, and Moshe Machover (1998). *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes*. Cheltenham, U.K.:Edward Elgar.

Leech, Dennis, and Robert Leech (2005). *Computer Algorithms for Voting Power Analysis*. University of Warwick. **http://www.warwick.ac.uk/~ecaae/** .

Miller, Nicholas R. (2008). "Voting Power with District Plus At-Large Representation." Presented at the 2008 Annual Meeting of the Public Choice Society, San Antonio, Texas.

Miller, Nicholas R. (2009). "*A Priori* Voting Power and the U.S. Electoral College" *Homo Oeconomicus*, 26 (3/4, 2009): 341-380.

Miller, Nicholas R. (2010). "Election Inversions by the U.S. Electoral College," paper based on a presentation at Voting Power and Procedures (VPP) Summer Workshop on Assessing Alternative Voting Procedures, Chateau du Baffy, Normandy, France, July 20-August 2, 2010.

Schlesinger, Arthur M., Jr. (2000). "Fixing the Electoral College." *Washington Post*, December 19, 2000, p. A39.