VOTING POWER WITH DISTRICT PLUS AT-LARGE REPRESENTATION

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Abstract

In an influential paper, Edelman (2004) argues that, compared to the situations in which all members of a legislative assembly are elected from single-member districts or all members are elected at-large, the indirect (Banzhaf) voting power of individual citizens can be increased by an electoral scheme that mixes district and at-large representation. Edelman further shows that this power is maximized when the number of members elected at large is equal to the square root of the total assembly size. An explicit and key assumption in Edelman's analysis is that voters casts separate and independent votes for district and at-large representation. However, if we stay within the standard voting power framework based on binary outcomes, voters never have reason to cast differing votes at the two levels. On the other hand, if elections have multifaceted outcomes that might justify casting differing votes, standard voting power analysis based on the Banzhaf measure does not straight-forwardly apply. In any event, Edelman's approach certainly does not apply to situations in which each citizen casts a single vote that counts (the same way) at both the district and at-large level, as exemplified by the Modified District Plan and the National Bonus Plan variations on the U.S. Electoral College. The major part of this paper is devoted to calculating individual indirect voting power under these plans. The analysis employs large-scale simulations of Bernoulli (random voting) elections.

Paul H. Edelman, "Voting Power and At-Large Representation," *Mathematical Social Sciences*, 47/2 (2004).

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As a warm-up exercise, let us consider the simplest case in which nine voters are partitioned into three districts. Elections are held under four distinct voting rules, each of which is symmetric with respect to both voters and two candidates A and B. With the U.S. Electoral College in mind, we sometimes call first-tier votes "popular votes" and second-tier votes "electoral votes." These are the four voting rules:

(1) *Pure District System*: there is 1 electoral vote for each district, and the candidate winning a majority of electoral votes (2 out of 3) is elected;

(2) *Small At-Large Bonus System*: there is 1 electoral vote for each district plus 1 at-large electoral vote, and the candidate winning a majority electoral votes (3 out 4) is elected (ties may occur in the second tier);

(3) *Large At-Large Bonus System*: there is 1 electoral vote for each district plus a bloc of 2 at-large electoral votes, and the candidate winning a majority of electoral votes (3 out 5) is elected; and

(4) *Pure At-Large Vote System*: there are 4 or more at-large electoral votes, so the candidate winning a majority of the popular votes (5 out of 9) is elected.

We consider things from the point of view of a focal voter *i* in District 1, who confronts $2^8 = 256$ distinct combinations of votes by the other eight voters. We want to determine, for each voting rule, in how many of the 256 combinations voter *i* is *decisive*, in the sense that *i*'s vote tips the election outcome for one way or the other.[1] The number of such combinations is voter *i*'s *Banzhaf score*, and the number of such combinations divided by 256 is voter *i*'s (absolute) *Banzhaf voting power*, in the two-tier voting game. If each distinct combination is equally likely (being generated by the *Bernoulli model* in which everyone votes randomly, i.e., as if independently flipping fair coins), voter *i*'s Banzhaf power is equal to the *probability* that *i* casts a vote that is *doubly*

[1] If the number of voters *n* is even (e.g., $n = 100$), the *interpretation* of a decisive vote differs somewhat according to whether the voting context is parliamentary or electoral. In a parliamentary body, a tie vote typically defeats a motion, so voter *i* decisive in any voting combination in which 50 other voters vote "yes" and 49 vote "no," as the motion passes or fails depending on whether *i* votes "yes" or "no." However, in elections between two candidates (our present concern), the voting rule is typically symmetric between the candidates, so a tie outcome might be decided by the flip of a coin. In this event, a voter *i* is "half decisive" in any voting combination in which 50 other voters vote for A and 49 for B (A wins if *i* votes for A and each candidate wins with .5 probability if *i* votes for B) *and also* in any voting combination in which 49 other voters vote for A and 50 for B. The upshot is that voter *i*'s total Banzhaf score (and voting power) is the same under either interpretation. Thus we can (and will) speak loosely "the probability of a tie vote" even when the number of voters is even. More obviously, we can (and will) speak interchangeably between "the probability of voter *i* breaking what would otherwise be a tie vote" and "the probability of a tie vote" when the number of voters is large.

decisive, i.e., the individual vote is decisive in *i*'s district and/or at-large *and* the district vote and/or the at-large bloc is decisive in the second tier.

Table 1 lists all 256 possible district vote profiles, shows the number of distinct voting combinations giving rise to each profile, indicates for each whether voter *i*'s vote is decisive under each of the four rules, and reports voter *i*'s Banzhaf score and voting power for each rule. We see that Banzhaf voting power increases as the weight of the at-large component increases.[2] The bottom of the table shows Banzhaf voting power defined in the manner of Edelman (2004), on the assumption that voters cast *separate and independent votes* at the district and at-large levels. In the Edelman setup, individual voting power is maximized with a mixture of district and at-large electoral votes such that the at-large component is approximately the square root of the total number of electoral votes. However, the voting game in Edelman's setup is not an ordinary simple voting game, and his voting power scores cannot be derived directly from Table 1.[3]

Table 2 is derived from Table 1 and has two types of entries in each cell. Table 2 crosstabulates the 256 voting combinations with respect to whether the vote in voter *i*'s district is tied, thereby making *i*'s vote decisive within the district (column variable), and whether the at-large vote is tied, thereby making *i*'s vote decisive with respect to the at-large vote (row variable). Each cell is called a *contingency*, and the lower number in each cell indicates number of voting combinations giving rise to each contingency. The contingencies themselves pertain to characteristics of the first-tier vote only. However, the four top numbers in each cell pertain to the second-tier voting rules and indicate the number of combinations in which *i*'s vote is doubly decisive vote, with a separate number for each of the four voting rules.

The numbers in Table 2 were determined by consulting Table 1, and Table 1 in turn was easy enough to construct. But if the number of voters expands even slightly, it becomes impossible to replicate Table 1 (for example, with 25 voters the number of possible combinations facing vote *i* is 2^{24} = 16,777,216), so some less direct method for enumerating (or estimating) Banzhaf scores and voting power values must be devised. The rest of this note is devoted to outlining such a method, which we illustrate by means of an example in which there are n =100,035 voters uniformly partitioned into k = 45 districts, each with a single electoral vote and with a bloc of 6 additional electoral votes elected at-large.[4]

We note two relevant baselines. Given 51 districts (with 1961 voters in each) and no at-large seats and using the standard approximation $\sqrt{2 / \pi n}$ with n = 1961 for the probability of a tie vote

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[2] However, with only nine voters, the Large At-Large Bonus System is effectively equivalent to the National Popular Vote system, because the candidate who wins the at-large vote must win at least one district and thus 3 out of 5 electoral votes.

[3] In the Edelman setup, any district vote profile nay occur with any popular vote split and, in particular, a candidate can win the at-large vote without carrying any district.

[4] In Edelman's setup, 44 districts plus 7 at-large votes maximizes individual voting power, since 7 is the integer closest to the square root of the total number of 51 electoral votes.

in a district, individual voting power within a district is .0180178. Using Leech's *Voting Power Algorithms* (or any similar facility), the voting power of each district in the second tier is .112275. Thus the individual two-tier voting power (the probability of double decisiveness) is .0180178 × .112275 = .00202295. At the other extreme, with 25 or fewer districts (i.e., direct popular vote), individual two-tier voting power is $\sqrt{2 / \pi n}$ with n = 100,035 or .00252269.

We begin with Table 3A, set up in the same manner as Table 2 and initially pertaining to first-tier votes only. Since the actual number of voting combinations is impossibly large, it displays *proportions* of combinations associated with each contingency or, equivalently under the Bernoulli model, the *probability* that each contingency will arise.

We first calculate the probability that the popular vote is tied, which gives us the total in the first row. As noted just above, this probability is about .00252269. Using the same approximation with n = 100,035/45 = 2223, we calculate the probability that the vote in i's district is tied to be .0169227, which gives us the first column total. Subtraction from 1.000000 gives us the totals in the second row and second column (Table 3A).

So far as Edelman-style calculations are concerned, we are close to done. If district and at-large votes are separate and independent, we can calculate the probabilities of contingencies simply by multiplying the corresponding row and column probabilities, as shown in Table 3B. But, given Edelman's assumptions, we need not be concerned with the interior cells at all. We need look only at the totals for the first row and first column and then take account of voting at the second tier. Second-tier voting is given by the voting rule 46(26:6,1, . . . ,1) — that is, a weighted voting game with 46 players (45 districts plus the at-large bloc), a quota of 26 (a bare majority of the total of 51 electoral votes), and voting weights of 6 for the at-large bloc and 1 for each district in which each voting combination occurs with equal probability. Leech's *Voting Power Algorithms* facility) produces .628702 and .080083 as the voting power for the at-large bloc and each district respectively. The voting power of voter i through district representation is his probability of being decisive within his district times the probability that is district is decisive in the second tier, i.e., .0169227 × .080083 = .0013552, and i's voting power through at-large representation is his probability of being decisive in the popular vote times the probability that the at-large bloc is district is decisive in the second tier, i.e., .00252227 × .628702 = .0015860. Within Edelman's setup, the overall voting power of each voter is simply the sum of these probabilities, i.e., .0029412 (Table 3C).[5] Note that this is greater than voting power under direct popular election, i.e., .0025227. Chart 1 shows Edelman-style voting power for all magnitudes of at-large representation.

If, in contrast to the Edelman setup, each voter has a single vote that counts the same way for both district and at-large representation, the resulting voting game is an ordinary simple voting game,

[5] Taking the sum of the voting powers associated with each of the voter's (district and at-large) votes may appear to double-count those voting combinations in Contingency 1 in which *both* of i's two votes are doubly decisive, but at the same time it misses voting combinations in Contingency 1 in which *neither* vote by itself is doubly decisive but the two votes together are, and it turns out that these combinations exactly balance out (Beisbart, 2007).

and individual two-tier voting power cannot exceed the .0025227 level resulting from direct popular(Pure At-Large) election.[6] However, voting power calculations become far more complex.

We first return to Table 3B and observe that, in the single-vote setup, the marginal probabilities are the same. However, the fact that voters cast the same vote for both district and at-large representation induces some correlation between the vote in any district and the at-large vote, so the probability that both votes are tied is greater than in the Edelman setup.

We can directly calculate the *conditional probability* that the at-large vote is tied given that the district vote is tied. Given that the vote in i's district is tied, the overall at-large vote is tied if and only if there is also a tie in the residual at-large vote after the votes cast in voter i's district are removed. The probability of this event is given by the standard approximation $\sqrt{2 / \pi n}$, where n is now 100,035 − 2223 = 97812, and is equal to .0025512 (Table 4A). We can now derive the unconditional probability that both types of ties occur simultaneously by multiplying this conditional probability by the probability that the district vote is tied in the first place, i.e., .0025512 × .0167366 = .0000432. With this piece of the puzzle in place, the probabilities of the other contingencies are also determined (Table 4B). Comparing Tables 4B and 3B, we observe that the probabilities of the contingencies differ only minutely, so the substantially lower overall voting power arising from this setup relative to Edelman's results almost entirely from the workings of second-tier voting.

In any event, voter i is decisive in the two-tier voting process only if the at-large and district votes are both tied (Contingency 1), the at-large vote only is tied (Contingency 2), or the district vote only is tied (Contingency 3). Having determined the probabilities of these contingencies, our next — and much more difficult — task is to determine, given each of these contingencies, the probability that voter i's vote is decisive in the second tier as well.

First, let's form some general expectations. Contingency 1, being the conjunction of two already unlikely circumstances, is extraordinarily unlikely occur; but, if it does occur, voter i is very likely to be doubly decisive. Voter i is doubly decisive if and only if neither candidate has won a majority of 26 electoral votes from the 44 other districts — put otherwise, if each candidate has won between 19 and 25 districts. By breaking a tie in both his district and at-large vote, voter i is tipping 7 electoral votes one way or the other and thereby gives one or other candidate the 26 electoral votes required for election. Given the Bernoulli model, the electoral votes of the other 44 districts are likely to be quite evenly divided, so it likely that neither candidate has won 26 districts.

At this point, it may be tempting to note that the second tier voting game is 45(26:7,1,...,1) and that the Banzhaf voting power of the at-large plus one district bloc of 7 votes is .708785, and from this to conclude that voter i's probability of double decisiveness through Contingency 1 is therefore .0000432 × .708785 = .0000306. However, to interpret the Banzhaf voting power of the at-large plus one district bloc calculated in this manner as the probability of second-tier decisiveness is to assume that all second-tier voting combinations are equally likely and, in particular, are independent of votes at the district level (as they are in the Edelman setup). But, given that the district and at-large votes are the same vote counting the same way, this assumption is not justified.

6 See Felsenthal and Machover (1998), pp. 58-59.

Contingency 2 is much more likely to occur than Contingency 1, while voter *i*'s probability of double decisiveness is only slightly less. Voter *i* is now doubly decisive if and only if neither candidate has won a majority of 26 electoral votes from all 45 districts — put otherwise, if each candidate has won between 20 and 25 districts. By breaking an at-large vote tie, voter *i* is tipping 6 electoral votes one way or the other and thereby gives one or other candidate the 26 electoral votes required for election. Again, given the Bernoulli model, the electoral votes of the 45 districts are likely to be quite evenly divided, so it quite likely that neither candidate has won 26 districts.

At this point, it may again be tempting to note that the second tier voting game is 46(26:6,1,...,1) and that the Banzhaf voting power of the at-large bloc is .628702, and from this to conclude that voter *i*'s probability of double decisiveness through Contingency B is therefore .0024800 × .628702 = .0015592. But again this assumes that all second-tier voting combinations are equally likely, and again this assumption is not justified.

Contingency 3 is still more likely to occur than Contingency 2, but voter *i* is far less likely to be doubly decisive in this contingency. Voter *i* is doubly decisive if and only if neither candidate has won a majority of 26 electoral votes from the other 44 districts *and* the at-large bloc of 6 votes, i.e., in the event that there is a overall 25-25 electoral vote tie. Such a tie results if and only if one candidate has carried 25 districts, while the other candidate has carried 18 districts *and* the at-large vote. The probability of such an event is small for three reasons:

(1) an *exact* tie in the second-tier electoral vote tie is required, because *i* is tipping only a single electoral vote;

(2) the split in *district* electoral votes must be unequal in a degree that depends on the number of at-large seats (25 to 19 with 6 at-large seats) in order to create a tie in overall electoral votes, and such an unequal split is less likely than an equal split, since Bernoulli elections always produce 50-50 expectations; and

(3) this rather unlikely 25-19 split in favor of one candidate in terms of district electoral votes must come about in the face of a popular vote majority in favor of the *other* candidate.

The last point implies that, in Contingency 3, voter *i* is doubly decisive only if *i*'s vote can bring about the kind of *election reversal* (or "reversal of winners," "wrong winner," "election conflict," "referendum paradox," etc.) in which the candidate who wins with respect to district (but excluding at-large) electoral votes at the same time loses with respect to the overall at-large (popular) vote. It is characteristic of districted election systems such as U.S. Presidential elections and U.K. general elections that such election reversals may occur, but they are generally thought to be quite unlikely unless the (at-large/popular vote) election is very close. However, we must bear in mind that almost all large-scale Bernoulli elections are extremely close. Indeed, if district and at-large votes are cast separately and independently in the Edelman manner, it is evident that 50% of all Bernoulli elections will entail election reversals. This is shown in Chart 2, which is based on a sample of 30,000 simulated elections in which the at-large vote was generated independently of the district vote. In contrast, if the popular vote is the district vote summed over all districts, a substantial correlation is induced between district and at-large votes, which considerably reduces the incidence of election reversals. This is shown in Chart 3, which is based on the same simulated elections but with the at-large vote being the sum of the district votes.[7]

Having formed expectations about the probability of double decisiveness in each contingency, we must now make actual calculations. While it may be possible to proceed analytically, I have found the obstacles to be formidable and have instead proceeded on the basis of large-scale simulations. For the present case with 45 districts and 6 at-large seat, I have generated four samples of 300,000 Bernoulli elections, for a total of 1.2 million cases.[8]

The next question is how to use the results of these simulations to estimate the relevant probabilities. The most direct approach is to simply produce the crosstabulation depicted in Tables 2-4. Table 4C shows the absolute frequencies resulting from these simulations. The number in the lower part of each cell is the number of times that contingency arose. The number in the upper part of each cell is the number of times voter *i*'s was doubly decisive in that contingency. Overall, voter *i* was doubly decisive in 2970 elections out of 1,200,000. Thus the estimated *a priori* voting power of voter i (and every other voter as well, given the overall symmetry) is 2970/1,200,000 or .002475. Our confidence in this estimate is reinforced by comparing Table 4D, in which all absolute frequencies in Table 4C are converted into proportions (and estimated probabilities), with Table 4B. It is evident that the simulation data closely matches the expected probabilities of each contingency calculated earlier.

A second approach is to replace the estimated probabilities of each contingency in the lower part of each cell in Table 4D with the known probabilities displayed in Table 4B. In this case, the numbers are so similar that this substituion makes essentially no difference. (Voter *i*' estimated voting power becomes .002488.)

A third approach is suggested if we examine the frequency distributions underlying Table 4C. Chart 4 shows the frequency distribution of districts won by Candidate A in the 51 elections in which both the district and at-large votes are tied. Voter *i* is doubly decisive provided that the number of districts won by either candidate lies in the range of 19-25. This was true in 49 elections out of the 51 elections, giving voter *i* a .960785 probability of double decisiveness in this contingency. But it evident that another sample of 1.2 million Bernoulli elections (including about 50 belonging to Contingency 1) could produce a rather different statistic. Given the existing sample, a more reliable

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7 Large-scale simulations run by Feix et al. (2004) found that, given Bernoulli (or "Impartial Culture") elections with 20 or more districts, election reversals occur in about 20.5% of all voting combinations. In Chart 3, election reversals occur 20.4% of the time. It is also clear that, in contrast to the Edelman setup, the probability of reversals declines as the election outcomes become even slightly lopsided.

8 Each of the 45 districts has 2223 voters, a number selected so that both district and at-large vote ties may occur before focal voter *i* (in District 1) casts his vote and so that no ties occur after *i* has voted. The simulations, which are generated by SPSS syntax files, operate at the level of the district: the vote for candidate A in each district is a number drawn randomly from a normal distribution with a mean of 2223/2 = 1111.5 and a standard deviation of $\sqrt{.25 \times 2223}$ and then rounded to the nearest integer.

estimate of voter *i*'s probability of double decisiveness can be derived by supposing that the distribution of districts won be Candidate A is normally distributed with a known mean of 22 (i.e., one half of the 44 districts other than voter *i*'s), rather that the sample statistic of 22.294118, and with a standard deviation of 2.032674. (From this point of view, the main purpose of the simulation is to provide an estimate of this standard deviation.) The estimated proportion of times voter i is doubly decisive is therefore equal the proportion of the area under a normal curve that lies within 3.5/2.032674 = 1.72187 standard deviations from the mean, which is .914907. This suggests that the direct result of the simulation of .960785 is too high.

In like manner, Chart 5 shows the frequency distribution of districts won by Candidate A in the contingency that the at-large vote only is tied. The actual distribution closely matches a normal with a mean of 22.5 (i.e., one half of all the 45 districts). Given the much large sample of elections in Contingency 2, it is unsurprising therefore that using the normal curve approach to estimating voter *i*'s double decisiveness produces almost the same result (.859592) as the sample statistic itself (.862838).

Chart 6 shows the frequency distribution of districts won by Candidate A in the contingency that the district vote only is tied. Since the distribution is clearly bimodal (which results from the fact that the at-large vote is not tied as in Charts 4 and 5 and one or other candidate has won the block of 6 at-large votes) and not normal, we clearly cannot use the normal curve approach. Moreover, Contingency 3 is by far the most likely of the three contingencies that allow voter *i* to be doubly decisive, so the sample statistic (367/20,167 = .01819805) is itself quite reliable.

Putting this altogether, using the normal curve approach to estimate probabilities of double decisiveness in Contingencies 1 and 2 and the sample statistic in Contingency 3 and the known probabilities for the contingencies themselves, we get an estimate of voter i's voting power of .002478, compared with .002475 using sample statistics only and .002488 using the sample statistics for probabilities of decisiveness in conjunction with the known probabilities for the contingencies themselves. In sum we can be pretty confident that the true value of voter *i*'s voting power is a bit under .0025, putting it slightly but pretty clearly below the value of .002523 that results from direct popular vote. This of contrasts of the Edelman value of .0029412 that results when voters cast separate and independent votes at the district and at-large levels.

Comparing Charts 4-6 with the Charts 7-9 that result in the Edelman setup makes evident how the Edelman setup produces a greater probability of double decisiveness. We see that each contingency occurs with essentially the same probability in the two setups (as we saw theoretically in the calculations displayed in Tables 3B and 4B). And in the first two contingencies, the voter is actually less likely to be doubly decisive in the Edelman setup as the spread in districts won by either candidate is substantially larger. This results from correlation between popular votes won and number of districts won that results when each voter casts a single vote that counts twice (Chart 3) rather that two separate and independent votes (Chart 2). But this effect is more than wiped out in Contingency 3, where two setups result in quite different distributions of electoral votes won. In the single-vote setup, the distribution is strikingly bimodal (the distance between the modes depending on the number of at-large electoral votes relative to the total) because, as a candidate wins more districts, he is more likely to win the at-large vote as well, whereas in the Edelman setup no such correlation exists. Given the parameters we are working with (6 at-large electoral votes out of 51), the Edelman setup produces a distribution that is unimodal but, relative to a normal curve, slightly "squashed" in the center. If the relative magnitude of the at-large component were increased, the "squashing" effect would be increased and would in due course produce bimodality, but it would always be substantially less than in the single-vote setup with the same at-large component. Thus, unless at-large component is wholly controlling (e.g., 26 electoral votes out of 51), the Edelman setup makes a even split of total electoral votes far more likely than does the single-vote setup and thereby greatly enhances the probability of double decisiveness in Contingency 3, which in turn is by far the probable contingency that (in either setup) allows double decisiveness.

I have duplicated the same kinds of simulations, but with fewer than 1.2 million elections, for other odd values of the at-large component within a fixed total of 51 electoral votes. The results are displayed in Chart 10. Simulations were run in blocks of 300,000 elections, and the voting power estimates are displayed individually (along with the means) for each at-large magnitude. It is evident that even blocks of this size produce considerable sampling error, but the general pattern of the relationship between the magnitude of the at-large component and individual power is very clear and in sharp contrast with the pattern of the same relationship in the Edelman setup (Chart 1).

To this point, we have considered only voting systems with uniform districts — that is, such that all districts have the same number of voters and electoral votes. But the most obvious application of this kind of analysis is to variants of the U.S. Electoral College — in particular, the National Bonus Plan and the Modified District Plan (already used by Maine and Nebraska) — in which the districts (i.e., the states) are far from uniform. Using estimating procedures generally similar (but necessarily more burdensome and with samples of 256,000 Bernoulli elections for each magnitude of the national bonus), to those outlined above for the uniform case, Chart 11 displays individual voting power in each state as the magnitude of the at-large component increases. Sampling error probably accounts for the minor anomalies in the chart, but again the overall pattern is clear enough. Chart 11 may be contrasted with Chart 12, which is based on the Edelman setup (and, being based on essentially exact calculations, is free of anomalies).

References

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All Possible Vote Profiles (Bipartitions) Confronting a Focal Voter *i* in District 1, Given a Total of Nine Voters Uniformly Partitioned into Three Districts

			Number of Times Voter i is Decisive (i's Bz Score)			
Pop. Vote	***District Vote Profile***	***n****	DV	DV + 1 AL	DV + 2 AL	PV (all AL)
8-0	(2-0) (3-0) (3-0)	1	0	0	0	0
	Total	**1**	**0**	**0**	**0**	**0**
7-1	(1-1) (3-0) (3-0)	2	0	0	0	0
	(2-0) (2-1) (3-0)	3	0	0	0	0
	(2-0) (3-0) (2-1)	3	0	0	0	0
	Total	**8**	**0**	**0**	**0**	**0**
6-2	(0-2) (3-0) (3-0)	1	0	0	0	0
	(1-1) (2-1) (3-0)	6	0	0	0	0
	(1-1) (3-0) (2-1)	6	0	0	0	0
	(2-0) (3-0) (1-2)	3	0	0	0	0
	(2-0) (2-1) (2-1)	9	0	0	0	0
	(2-0) (1-2) (3-0)	3	0	0	0	0
	Total	**28**	**0**	**0**	**0**	**0**
5-3	(0-2) (3-0) (2-1)	3	0	0	0	0
	(0-2) (2-1) (3-0)	3	0	0	0	0
	(1-1) (3-0) (1-2)	6	6	3**	0	0
	(1-1) (2-1) (2-1)	18	0	0	0	0
	(1-1) (1-2) (3-0)	6	6	3**	0	0
	(2-0) (3-0) (0-3)	1	0	0	0	0
	(2-0) (2-1) (1-2)	9	0	0	0	0
	(2-0) (1-2) (2-1)	9	0	0	0	0
	(2-0) (0-3) (3-0)	1	0	0	0	0

	Total	**56**	**0**	**6****	**0**	**0**
4-4	(0-2) (3-0) (1-2)	3	0	1.5**	3	3
	(0-2) (2-1) (1-2)	9	0	4.5**	9	9
	(0-2) (1-2) (3-0)	3	0	1.5**	3	3
	(1-1) (3-0) (0-3)	2	2	2	2	2
	(1-1) (2-1) (1-2)	18	18	18	18	18
	(1-1) (1-2) (2-1)	18	18	18	18	18
	(1-1) (0-3) (3-0)	2	2	2	2	2
	(2-0) (2-1) (0-3)	3	0	1.5**	3	3
	(2-0) (1-2) (1-2)	9	0	4.5**	9	9
	(2-0) (0-3) (2-1)	3	0	1.5*	3	3
	Total	**70**	**40**	**55**	**70**	**70**
3-5	**Dual of 5-3**	**56**	**12**	**6**	**0**	**0**
2-6	**Dual of 6-2**	**28**	**0**	**0**	**0**	**0**
1-7	**Dual of 7-1**	**8**	**0**	**0**	**0**	**0**
0-8	**Dual of 8-0**	**1**	**0**	**0**	**0**	**0**
	Total [Bz Score]	**256**	**64**	**67**	**70**	**70**
	Bz Power		**.25**	**.26172**	**.27344**	**.27344**
	Edelman Bz Power* **		**.25**	**.29004**	**.33008**	**.27244**

* ***n*** is the number of distinct voter combinations giving rise to the specified district vote profile.

** In these profiles, Bz awards voter *i* "half credit" as *i*'s vote is decisive with respect to whether a particular candidate wins or there is a tie between the two candidates. (Under the other voting rules, ties cannot occur.)

*** Edelman Bz Power = $\frac{\text{Prob. i decisive}}{\text{in district}} \times \frac{\text{Prob. district}}{\text{decisive in Tier 2}} + \frac{\text{Prob. i decisive}}{\text{at-large}} \times \frac{\text{Prob at-large}}{\text{decisive in Tier 2}}$

AL = 1:	.5	×	.375	+	.27344	×	.375	= .29004
AL = 2:	.5	×	.25	+	.27344	×	.7	= .33008

TABLE 1

	DV Tied	DV Not Tied	Total
PV Tied	40 / 40 / 40 / 40 40	0 / 15 / 30 / 30 30	40 / 55 / 70 / 70 70
PV Not Tied	24 / 12 / 0 / 0 88	0 / 0 / 0 / 0 98	24 / 12 / 0 / 0 186
Total	64 / 52 / 40 / 40 128	0 / 15 / 30 / 30 128	64 / 67 / 70 / 70 256

All District / 1 A-L / 2 A-L / All AL

TABLE 2

	DV Tied	DV Not Tied	Total
PV Tied	**Contingency 1**	**Contingency 2**	.0025227
PV Not Tied	**Contingency 3**	**Contingency 4**	.9974773
Total	.0169227	.9830773	1.000000

TABLE 3A

	DV Tied	DV Not Tied	Total
PV Tied	.0000427	.0024800	.0025227
PV Not Tied	.0168800	.9805973	.9974773
Total	.0169227	.9830773	1.0000000

TABLE 3B

	DV Tied	DV Not Tied	Total
PV Tied	.0000427	.0024800	× .628702 = .0015860 .0025227
PV Not Tied	.0168800	.9805973	.9974773
Total	× .080083 = .0013552 .0169227	.9830773	.0029412 1.0000000

TABLE 3C

	DV Tied	DV Not Tied	Total
PV Tied	.0025512 ⇓ .0000432		.0025227
PV Not Tied			.9974773
Total	.0169227	.9830773	1.0000000

TABLE 4A

	DV Tied	DV Not Tied	Total
PV Tied	.0000432	.0024795	.0025227
PV Not Tied	.0168795	.9805978	.9974773
Total	.0169227	.9830773	1.0000000

TABLE 4B

	DV Tied	DV Not Tied	Total
PV Tied	49 51	2554 2960	2063 3011
PV Not Tied	367 20,167	0 1,176,822	367 1,196,989
Total	416 20,218	2554 1,179,782	2970 1,200,000

TABLE 4C

	DV Tied	DV Not Tied	Total
PV Tied	.00003323 .0000426	.0017581 .0024805	.0017913 .0025231
PV Not Tied	.00018283 .0166940	.0000000 .9808269	.00018283 .9974769
Total	.00021606 .0167366	.0017581 .9833264	.0019742 1.0000000

TABLE 4D

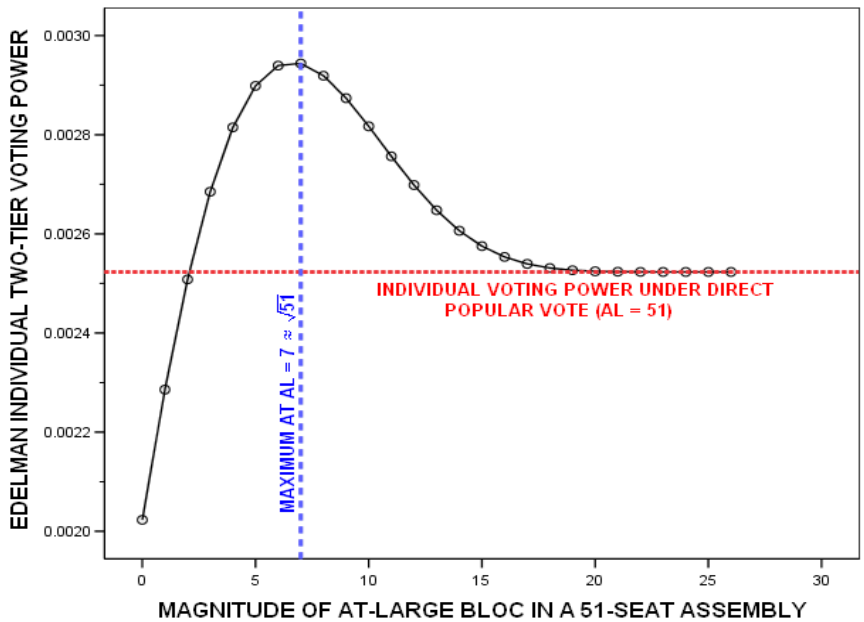


CHART 1

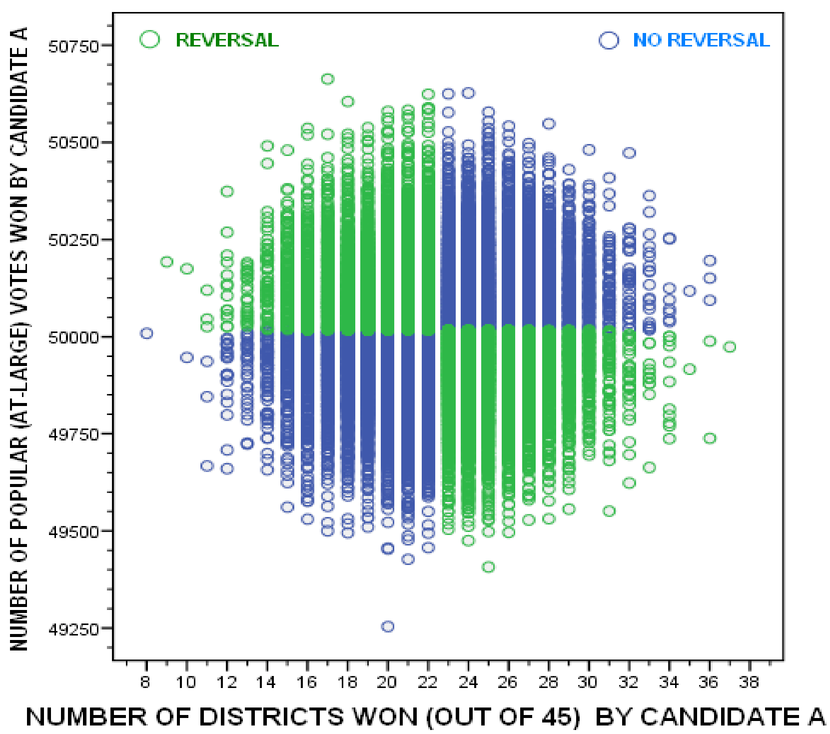


CHART 2

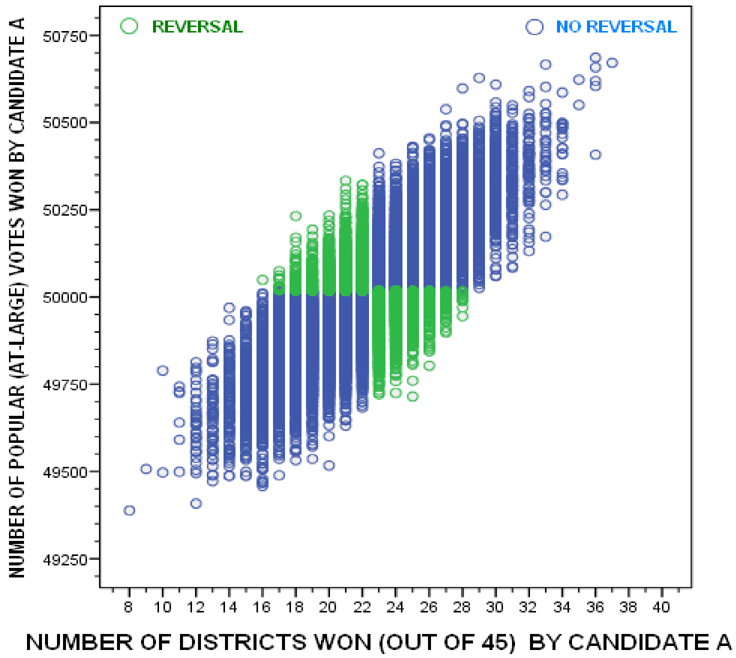


CHART 3

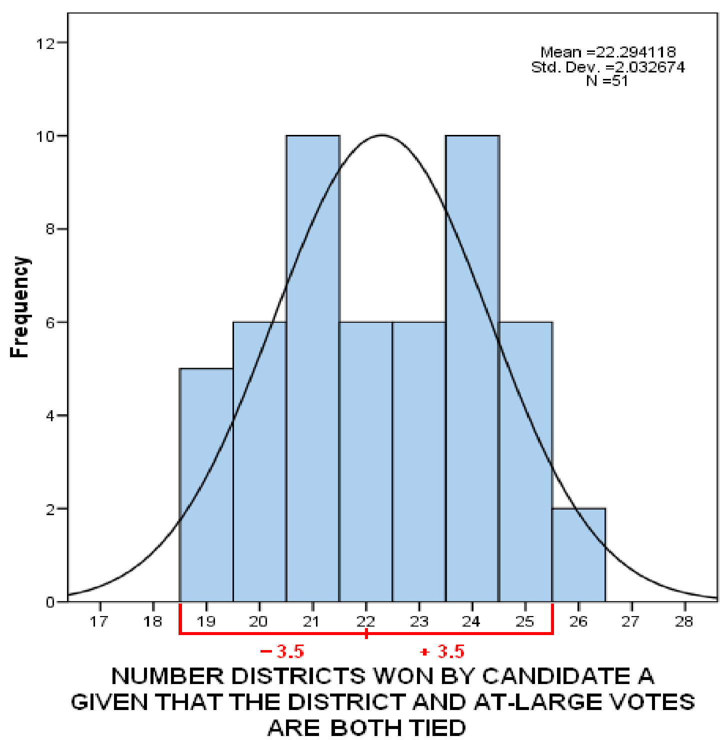
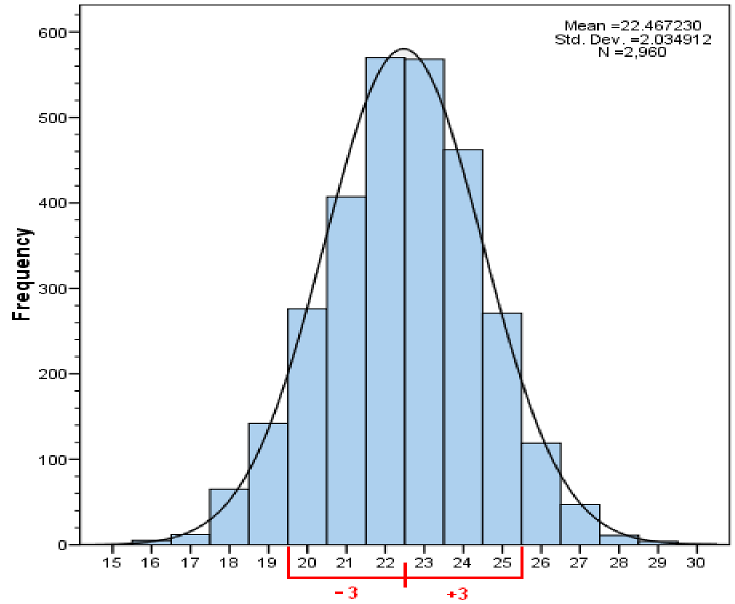
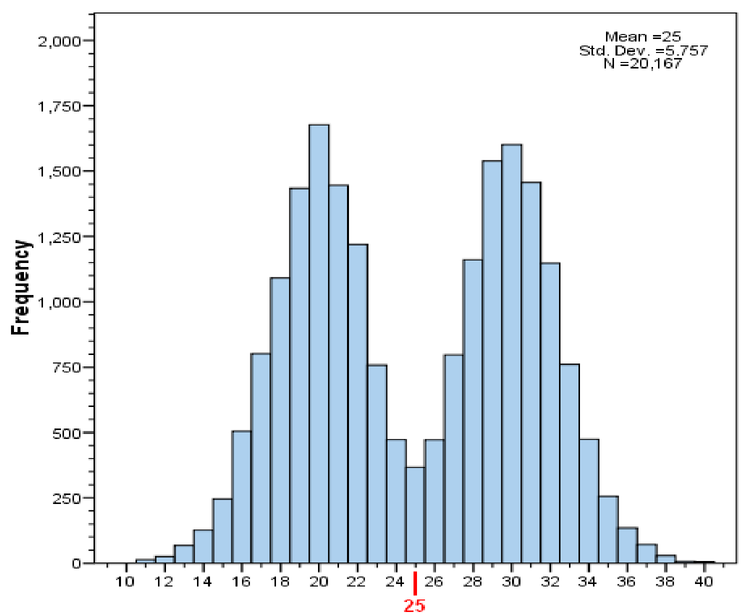


CHART 4



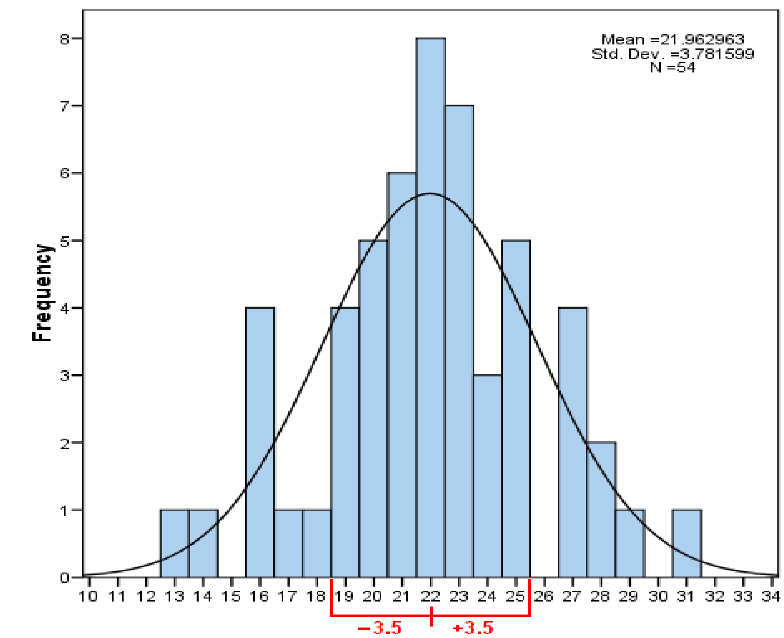
NUMBER OF DISTRICTS WON BY CANDIDATE A
GIVEN THAT THE AT-LARGE VOTE ONLY IS TIED

CHART 5



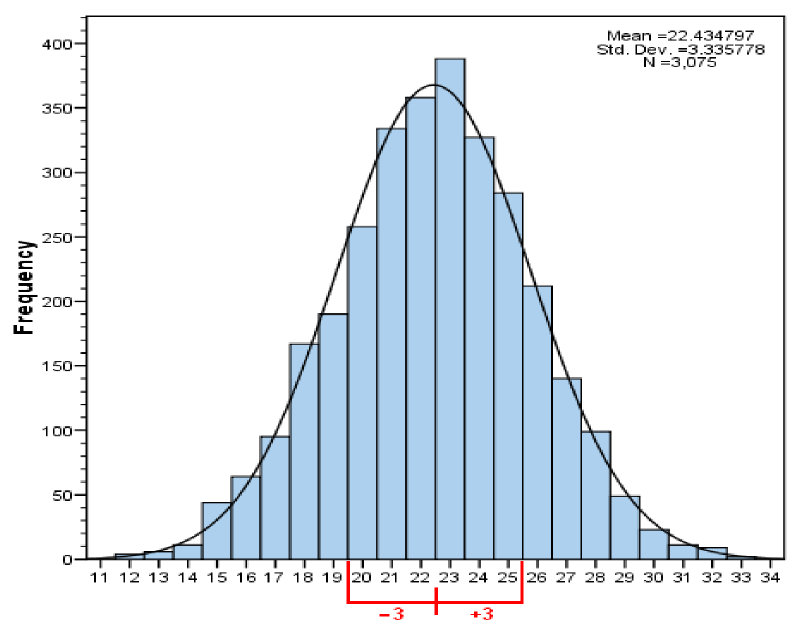
TOTAL NUMBER OF (DISTRICT + AT-LARGE) ELECTORAL
VOTES WON BY CANDIDATE A GIVEN THE THE DISTRICT
VOTE ONLY IS TIED

CHART 6



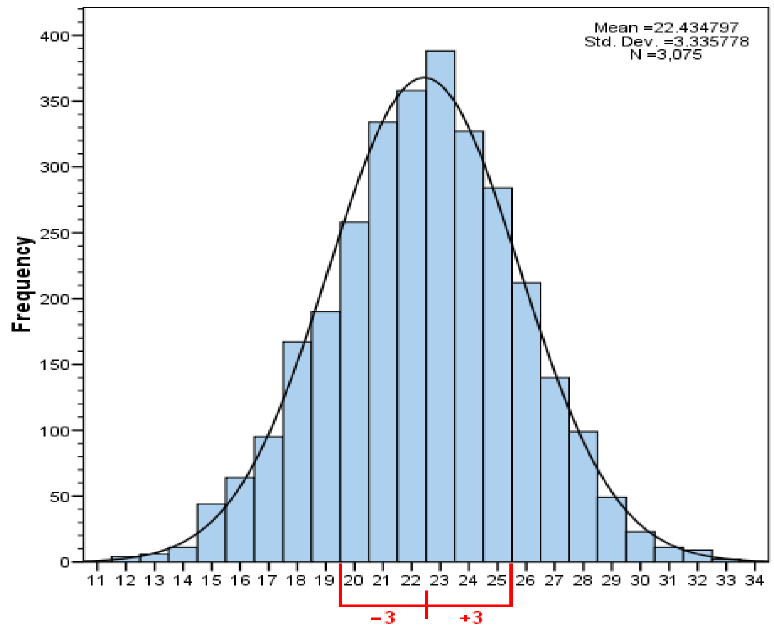
NUMBER DISTRICTS WON BY CANDIDATE A GIVEN THAT THE DISTRICT AND AT-LARGE VOTES ARE BOTH TIED (EDELMAN SETUP)

CHART 7



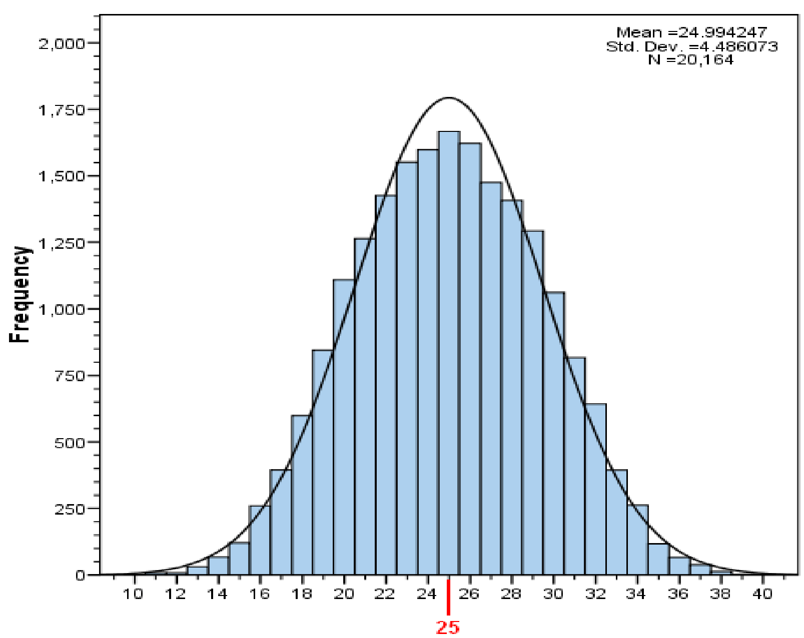
NUMBER OF DISTRICTS WON BY CANDIDATE A GIVEN THAT THE AT-LARGE VOTE ONLY IS TIED (EDELMAN SETUP)

CHART 8



NUMBER OF DISTRICTS WON BY CANDIDATE A
GIVEN THAT THE AT-LARGE VOTE ONLY IS TIED
(EDELMAN SETUP)

CHART 8



TOTAL NUMBER OF (DISTRICT + AT-LARGE) ELECTORAL
VOTES WON BY CANDIDATE A GIVEN THE THE DISTRICT
VOTE ONLY IS TIED (EDELMAN SETUP)

CHART 9

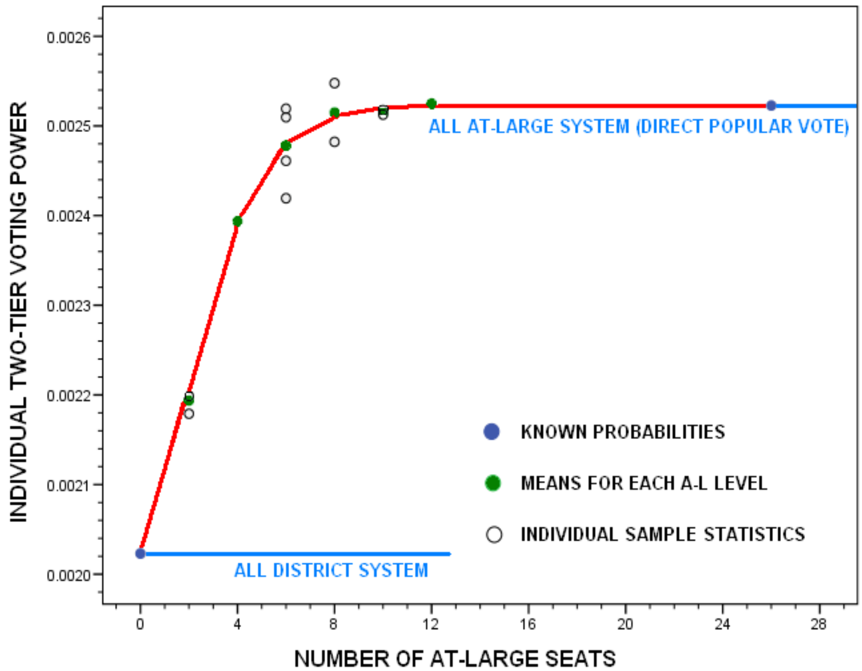


CHART 10

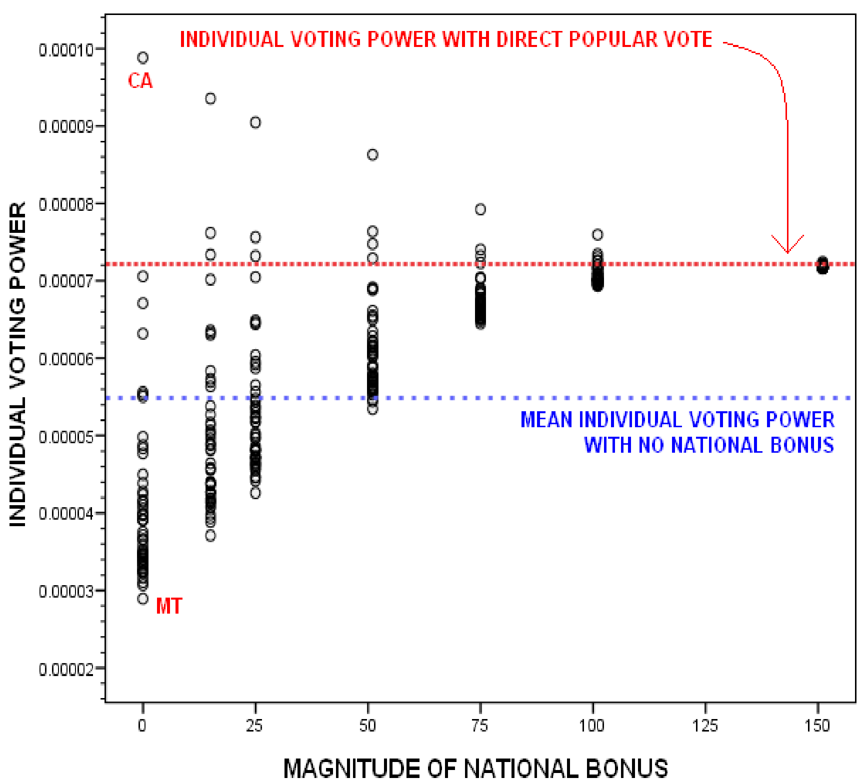


CHART 11

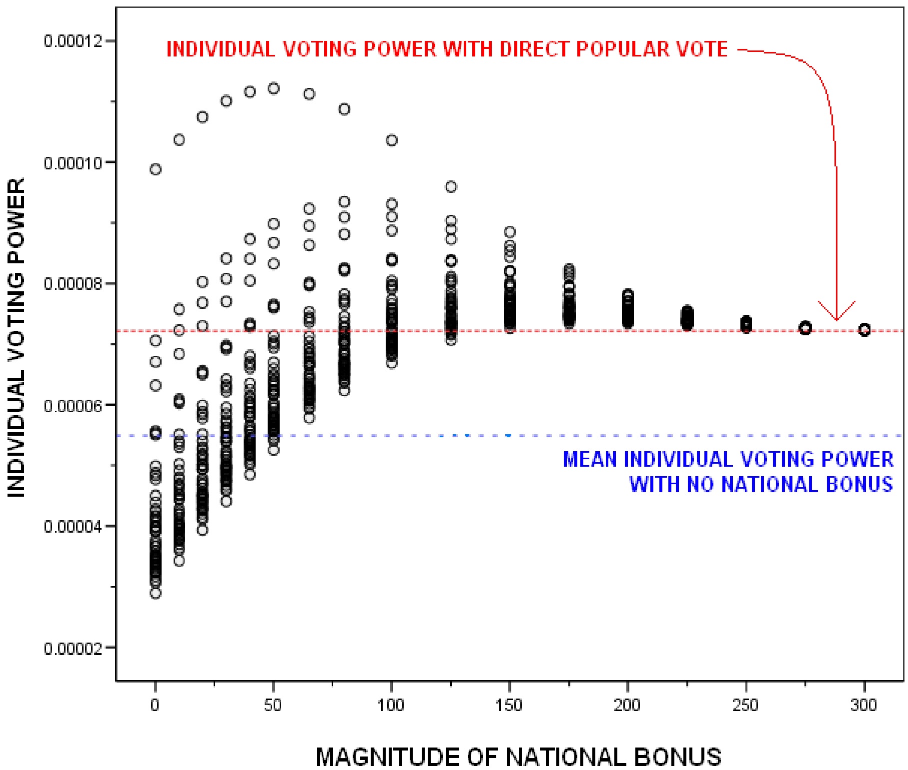

INDIVIDUAL VOTING POWER WITH DIRECT POPULAR VOTE
MEAN INDIVIDUAL VOTING POWER
WITH NO NATIONAL BONUS
INDIVIDUAL VOTING POWER
0.00012
0.00010
0.00008
0.00006
0.00004
0.00002
0
50
100
150
200
250
300
MAGNITUDE OF NATIONAL BONUS

CHART 12