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THE UNCOVERED SET IN SPATIAL VOTING GAMES

ABSTRACT. In a majority rule voting game, the uncovered set is the set of alternatives each of which can defeat every other alternative in the space either directly or indirectly at one remove. Research has suggested that outcomes under most reasonable agenda processes (both sincere and sophisticated) will be confined to the uncovered set. Most research on the uncovered set has been done in the context of voting games with a finite number of alternatives and relatively little is known about the properties of the uncovered set in spatial voting games.

We examine the geometry of the uncovered set in spatial voting games and the geometry of two important subsets of the uncovered set, the Banks set and the Schattschneider set. In particular, we find both general upper and lower limits on the size of the uncovered set, and we give the exact bounds of the uncovered set for situations with three voters. For situations with three voters, we show that the Banks set is identical to the uncovered set.

I. INTRODUCTION

In situations where there are cyclical majorities and therefore majority voting is indeterminate, recent research (e.g., Miller, 1980; McKelvey, 1983; Shepsle and Weingast, 1984; Banks, 1985; Miller *et al.*, 1985) has shown that the uncovered set of alternatives (first described by Miller, 1980) contains all of the outcomes that might arise from a variety of common group decision processes, including binary and multiple choice processes under both sincere and sophisticated voting. Because outcomes within the uncovered set arise with certainty or near certainty under virtually every agenda process ever discussed, properties of the uncovered set are of considerable importance. Until now, most of the consideration of the uncovered set has focused upon situations with finitely many alternatives; consequently, little is known about the geometry of the uncovered set in the spatial context. McKelvey (1983) provided some preliminary findings concerning the nature and size of the uncovered set in the spatial context; this paper provides tighter general bounds for the uncovered set, and precisely specifies it in the case of three voters. Also,

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we examine the geometry of two important subsets of the uncovered set, the Banks set and the Schattschneider set, and present some new results about each. To simplify our exposition we assume that (1) the number of voters is odd; (2) each voter can be identified with his ideal point, a point in $\mathbb{R}^2$; and (3) a voter prefers one alternative (point of $\mathbb{R}^2$) to another (different point of $\mathbb{R}^2$) if and only the first is closer to the voter's ideal point than is the second.

We show that the size of the uncovered set varies dramatically, depending upon the nature of the voter distribution; in some situations, it can be as large as the region bounded by the set of voter ideal points (the Pareto set); in other situations, it can be an infinitesimal proportion of the Pareto set. To the extent that we show that the uncovered set is frequently a small subset of the Pareto set, we will have shown that agenda processes are generally well behaved. To the extent that we can specify conditions under which the uncovered set will be a large subset of the Pareto set, we will be specifying the conditions under which majority rule processes in the spatial context may give rise to substantial indeterminacy. We will show both upper limits and lower limits of the uncovered set, and leave the interpretation of how big to observers in particular empirical contexts.

We express the upper and lower limits in terms of the yolk, a geometric construct whose radius is a measure of the degree of bilateral asymmetry that is present in a spatial array of voters. The yolk is a circle that is "centrally" located among the set of voter ideal points (Ferejohn *et al.*, 1984).

With any reasonable degree of bilateral symmetry in the distribution of ideal points, the yolk tends, for large sets of voters, to be a small subset of the Pareto set (McKelvey, 1983). Moreover, the yolk tends to shrink with the addition of voters (Feld *et al.*, 1985).

First we briefly review some important basic definitions. To aid the reader, we shall present our discussion in ordinary language rather than mathematical symbolism wherever possible.

DEFINITION 1. (a) The *win-set* of an alternative, y, is the set of alternatives that are majority preferred to y, and analogously (b) The *inverse win-set* of an alternative, y, is the set of alternatives to which y is majority preferred.

DEFINITION 2. An alternative, y, is *covered* by an alternative, z, if z is majority preferred to y and if, for every x to which y is majority preferred, z is also majority preferred to x; i.e., an alternative y is covered by an alternative z if the win-set of y contains the win-set of z.

DEFINITION 3. An alternative, y, is *uncovered* if there is no alternative that covers it.

We include an equivalent definition of "uncovered" that is useful for understanding and finding the uncovered set.

DEFINITION 3′ (Miller, 1980). An alternative, y, is *uncovered* if and only if it is majority preferred to all other feasible alternatives either directly or at one remove; i.e., for every z, either y is majority preferred to z, or there is an x such that y is majority preferred to x and x is majority preferred to z.

DEFINITION 4. The *uncovered set* is the set of all uncovered alternatives.

It is easy to see that points which are Pareto dominated, i.e., unanimously beaten by some point, cannot be in the uncovered set. Hence, the uncovered set is within the Pareto set.

We now proceed to specify bounds on the uncovered set in the spatial context.

II. MINIMUM BOUNDS ON THE UNCOVERED SET IN THE SPATIAL CONTEXT

We consider spatial situations where each point in a space represents an alternative that might be chosen by the majority of a group. Each individual in the group has an ideal point in the space and prefers alternatives that are closer to his/her ideal point to those which are further away. Such distance-based preference orderings are commonly assumed in modeling voting in the spatial context. We confine our consideration to two-dimensional situations for simplicity, but our results generally have straightforward extensions to higher dimensions.[1]

We begin by showing that we can easily find some alternatives that must be uncovered. First, we need some additional definitions.

DEFINITION 5. A *median line* is a line such that no more than half the voter ideal points lie on either side of it; essentially, it divides the set of voters in half. (None that each "half" contains less than half the ideal points because at least one ideal point is on the line itself).

Note that every line, ℓ, has exactly one median line perpendicular to it, and that the alternative at the point of intersection is majority preferred to every other point on the line ℓ.

DEFINITION 6. (a) A voter *projection* on a line is the intersection of the line with a perpendicular dropped to it from a voter ideal point; (b) the *median voter projection* on a line is the median such projection.

DEFINITION 7 (Grofman and Feld, 1985). For two dimensions, the *Schattschneider set* is the set of median voter projections on median lines; i.e., the set of alternatives that are at the perpendicular intersection of two median lines.

THEOREM 1 (Grofman and Feld, 1985). *For two dimensions, the Schattschneider set is a subset of the uncovered set.*

Proof. Using the alternate definition of an uncovered point, Definition 3′, we show that a median projection on a median line is directly or indirectly (at one remove) majority preferred to all alternatives. For any alternative, z, there is some alternative, x, on the median line that is majority preferred to z, i.e., the point at the projection of z on the line. Furthermore, the median voter projection on the median line, which we shall call y, is majority preferred to every point on the median line. Consequently, for every z, either there is an x such that y is majority preferred to x and x is majority preferred to z, and/or y is majority preferred to z. Hence y is uncovered. QED

Figure 1 shows the Schattschneider set for an example with three voters whose ideal points are located at (0, 0), (0, 1), and (1, 0). The Schattschneider set consists of the (heavily lined) cigar-shaped surface in

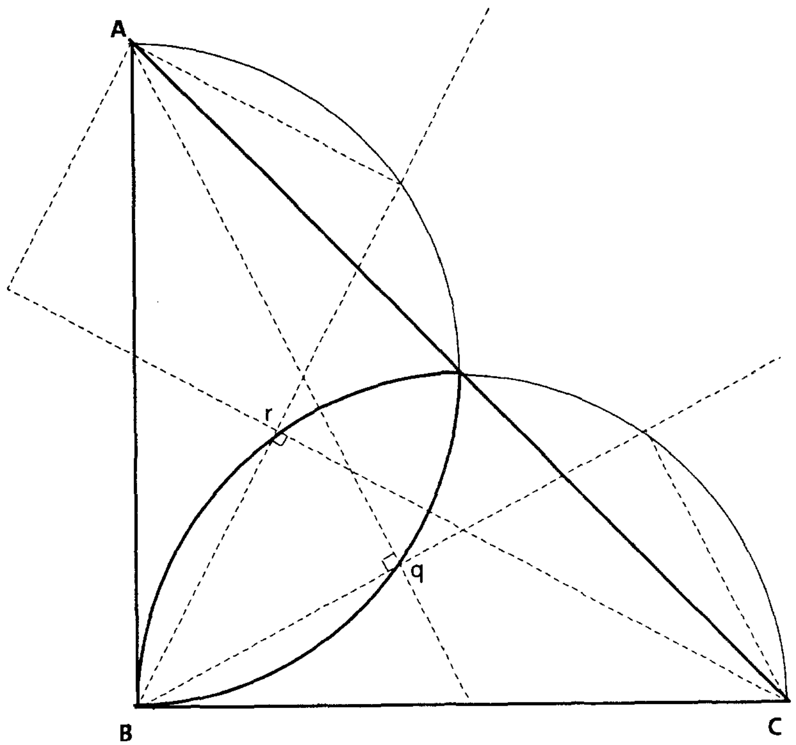

Fig. 1. Schattschneider Set for a right triangle.

Figure 1. In general the Schattschneider set consists of a collection of arcs of circles which together form a continuous closed curve (See Note 1).

The Schattschneider set is found by tracing out the perpendicular intersection of median lines. In a situation with three voters, the median lines are those lines that pass through a voter ideal point and either lie within the triangle formed by the three ideal points or pass through a second voter ideal point. In the right triangle in Figure 1, the angle at B is 90° so that one point of the Schattschneider set is B itself. More commonly, two median lines which are perpendicular will pass through exactly two voter ideal points. It is well known in geometry that the set of points at which lines through two given points intersect at right angles is the circle passing through the two given points and centered at the midpoint of the line joining them. Thus the Schattschneider set consists of the portions falling within the triangle of circles though two voter ideal points centered at their midpoint. A similar construction defines the Schattschneider set for more than three voters.

There are several other "centrally" located points in the Pareto set that must be uncovered. To discuss them we must introduce additional definitions.

DEFINITION 8. The *Copeland value* of a point, y, is the area of the win-set of y.

DEFINITION 9 (Grofman *et al.*, 1985; Copeland, 1951). The *strong point* (also known as the *Copeland winner*) is the point with the minimal Copeland value.

THEOREM 2 (Moulin, 1984; McKelvey, 1983). *The strong point is uncovered.*

Proof. If the strong point were covered, there would be a point whose win-set was within that of the strong point. That point would have a smaller win-set, contrary to the definition of the strong point. QED

In Figure 1, the strong point is (0.25, 0.25).

DEFINITION 10. The half win-set of a point is the set of points obtained by shrinking every ray from the point to the boundary of its win-set by a factor of one-half.

DEFINITION 11 (McKelvey, 1983; Ferejohn *et al.*, 1984). The *yolk* is the circle of minimal radius that intersects all median lines.

DEFINITION 11′ (Feld *et al.*, 1985). The *yolk* is the minimal circle surrounding the half win-set of a point.

Feld *et al.* (1985) showed that Definitions 11 and 11′ are equivalent.

THEOREM 3. *The center of the yolk is uncovered.*

Proof. If the center of the yolk were covered, there would be a point whose win-set was within that of the center of the yolk. The half win-set of that other point would be included within a circle smaller than the yolk, contrary to the definition of the yolk. QED

In Figure 1, the center of the yolk is at $(1-\sqrt{2}/2,\ 1-\sqrt{2}/2) \approx (0.293, 0.293)$

The uncovered set can consist of a single point or a rather oddly shaped region. A parameter which provides a measure on the size of the uncovered set is the radius of the smallest circle enclosing the uncovered set.

DEFINITION 12. The *uncovered circle* is the smallest circle that encloses all uncovered points.

COROLLARY 1 to THEOREM 1. *The uncovered circle includes a point on every median line, i.e., it intersects all median lines.*

Proof. As shown in Theorem 1, the median point on a median line is uncovered. Every median line thus has a median point that must be included in the uncovered set and thus must be included in the uncovered circle. QED

THEOREM 4. *The uncovered circle is at least as large as the yolk.*

Proof. The yolk is defined as the smallest circle intersecting all median lines. QED

THEOREM 5. *The uncovered circle overlaps the yolk. In particular, the uncovered circle encloses at least* $(4\pi-3\sqrt{3})/6\pi$ (≈ 0.39) *of the area of the yolk.*

Proof. Theorem 4 tells us that the uncovered circle has a radius at least equal to the radius of the yolk, and Theorem 3 tells us that it includes the center of the yolk. It can easily be seen that the least overlap under these conditions is achieved when the radius of the uncovered circle is the same as the yolk and the center of the uncovered circle is on the circumference of the yolk. Under these conditions, some straightforward geometry shows that the overlap is $(4\pi-3\sqrt{3})/6\pi$ of the area of the yolk. QED

THEOREM 6. *The points in any Von Neumann-Morgenstern* (V-M) *solution are uncovered.*

Proof. See McKelvey (1983)

In the case of a 3-voter game the unique V-M solution is the set of tangency points of the yolk with the boundaries of the Pareto set. (Recall that here the Pareto set consists of a triangle with voter ideal points as vertices). If we take the vertices of the triangle to be (0,1), (0,0) and (1,0), as in Figure 1, then the center of the yolk is as previously noted at $(1-\sqrt{2}/2,\ 1-\sqrt{2}/2) \approx (0.29,\ 0.29)$. The tangency points of the yolk are then $(0,\ 1-\sqrt{2}/2)$, $(1-\sqrt{2}/2,\ 0)$ and 1/2, 1/2). Note that only one of these points is a median point on a median line. Thus, knowing that the points in the V-M solution are in the uncovered set can provide us new information about where that set is located.

III. MAXIMUM BOUNDS ON THE UNCOVERED SET IN THE SPATIAL CONTEXT

The size and location of the uncovered set depends upon the exact distribution of voters and the resultant win-sets. In particular, it is easy to see that any point that is outside the win-set of every point in the win-set of a point r is covered by r. Furthermore, the set of points not covered by any point in some set S must contain the uncovered set. In particular, the set points not covered by any specific point s must contain the uncovered set.

We can determine general outer bounds on the uncovered set using information on the minimum and maximum bounds on the win-sets for all points that are a specified distance from the center of the yolk. We will show that the maximum win-set of a point a specified distance from the center of the yolk is contained within the minimum win-set of a point a specified greater distance from the center of the yolk in the opposite direction. Therefore, the inner point covers the outer point. Hence, we shall be able to find a distance such that all points which are at least that far from the center of the yolk must be covered.

To determine the minimum covering set for a point, we make use of the minimum and maximum win-sets for a point specified by McKelvey (1983) and Feld *et al.* (1985). Given the distance of a point from the center of the yolk, it is possible to specify the minimum and maximum distances that the win-set of that point can extend in any particular direction relative to the center of the yolk in terms of a cardioid.

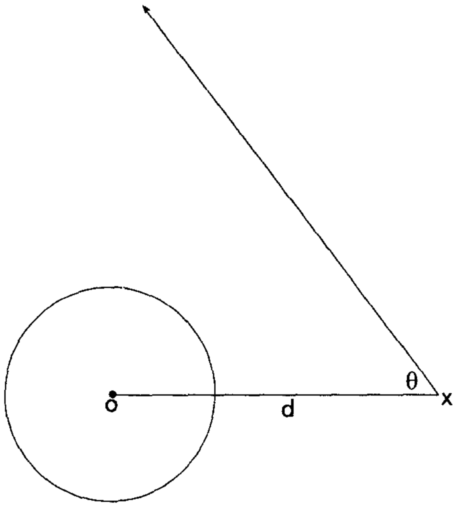

Fig. 2.

Let the point x be a distance d from the center of the yolk, o. Label as θ (theta) the angle between the line $\overline{ox}$ and a specified direction, as shown in Figure 2. Let r be the radius of the yolk.

THEOREM 6 (Feld *et al.*, 1985). *Maximum and minimum bounds on the size and direction of the win-set of any point can be given in terms of the yolk as follows:*

I. *Maximum bounds:*

(a) *If* $\cos\theta \leq -r/d$, *then* x *may be beaten by all points in the* θ *direction. (Note that if* $\theta > \pi/2$, *then* $\cos\theta < 0$*).*

(b) *If* $\cos\theta > -r/d$, *then* x *may be beaten by all points within* $2d\cos\theta + 2r$ *from* x *along the* θ *direction.*

The maximum bounds take the form of a (heart-shaped) "cardioid" around o, with the cusp at x and with positive eccentricity 2r.

II. *Minimum bounds:*

(a) If $\cos\theta \leq r/d$, *then* x *may be beaten by all points in the* θ *direction.*

(b) *If* $\cos\theta > r/d$, *then* x *may be beaten by all points within* $2d\cos\theta - 2r$ *from* x *in the* θ *direction.*

The minimum bounds take the form of a "fish-shaped" cardioid around o, with the cusp at x and with negative eccentricity $-2r$.

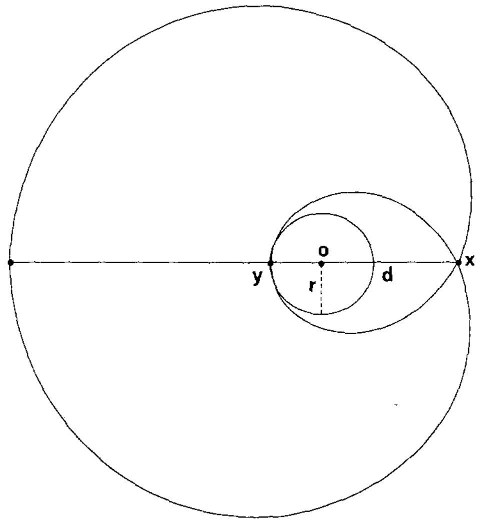

Fig. 3. Cardioid bounds on the win-set of x, $d=2.5r$. (Figures only approximate).

Proof. See McKelvey (1983), and Feld *et al.* (1985).
Figure 3 provides an illustration of the maximum and minimum win-sets of a point.

McKelvey (1983) has shown that the uncovered set is always contained within a circle centered at the center of the yolk and of radius $4r$, where r is the radius of the yolk. Using Theorem 6, we can improve upon this bound.

COROLLARY 1 to THEOREM 6. *A point* $3.7r$ *from the center of the yolk is always covered by a point* $0.4r$ *from the center of the yolk in the opposite direction.*

Proof. Computer calculation of the maximum win-set of $0.4r$ and the minimum win-set of the point $3.7r$ in the opposite direction shows that the maximum win-set of the first point is contained within the minimum win-set of the second; therefore, the first covers the second. QED

THEOREM 7. *All points outside of a circle* $3.7r$ *around the center of the yolk are covered; i.e., the uncovered set is contained within a circle of radius* $3.7r$ *around the center of the yolk.*

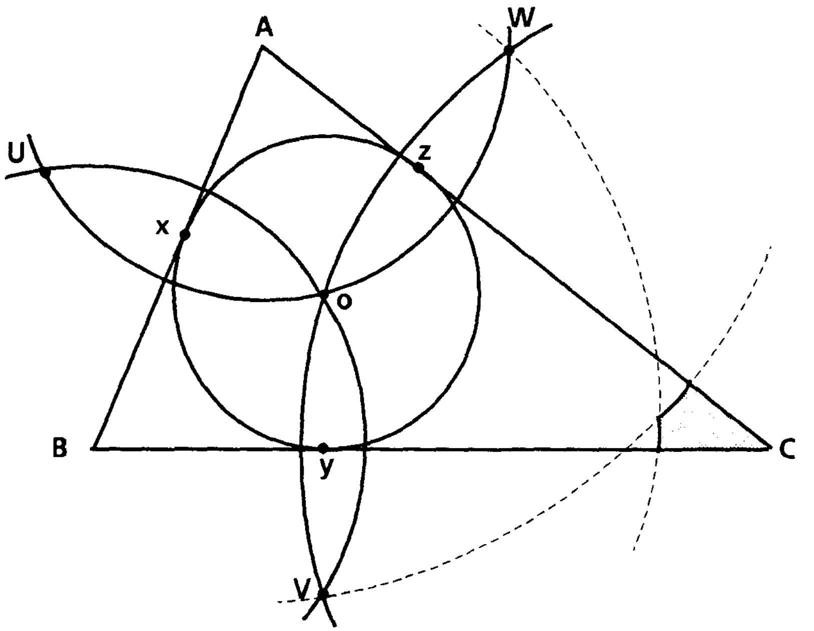

Fig. 4. Points in the shaded region are in the Pareto set but known to be covered by the center of the yolk.

Proof. From Corollary 1 to Theorem 6, the maximum win-set of a point $0.4r$ in any direction is contained within the minimum win-set of a point $3.7r$ in the other direction. But this minimum win-set is contained within the minimum win-sets of all points further out in that same direction.[2] Therefore, x covers those points as well. QED

The set of points covered by any point cannot be in the uncovered set.[3] In particular, the set of points not covered by the center of the yolk contains the uncovered set. Since all uncovered points lie in the Pareto set, the intersection of the Pareto set and the set of points not covered by the center of the yolk is also an upper bound for the uncovered set. Sometimes, this bound will be well below our general 3.7 radii limit. For a three voter situation, we show the construction of this set in Figure 4. Points in the shaded area cannot beat the center of the yolk either directly or indirectly and therefore must be covered. The construction used, based upon the win-set of the win-set of the center of the yolk, generalizes to any number of voters.

In the case of an arbitrary right triangle, a construction like that in Figure 4 can be shown to imply that the uncovered set lies within roughly $3.42r$ of the center of the yolk. For the isosceles right triangle, this construction merely gives the Pareto set as the bound.

For the case of three voters, it is possible to obtain an exact specification of the uncovered set. In general, the uncovered set is a proper subset of the win-set of the win-set of the center of the yolk. This specification and related results are given in the next section.

IV. OTHER BOUNDS FOR THE UNCOVERED SET

We wish to consider how large or small the uncovered set may be relative to the Pareto set, the region defined by the set of voter ideal points. When one alternative is a majority winner, then that alternative is the median point on all median lines and the yolk shrinks to that single point. In general, as we move closer toward a single majority winner, the yolk shrinks and the uncovered set becomes arbitrarily small relative to the Pareto set (Cox, 1985; McKelvey, 1985; Feld *et al.*, 1985). On the other hand, there are cases in which the uncovered set is the entire Pareto set; specifically, when there are three voters with ideal points arranged in an equilateral triangle, the uncovered set is the entire triangle, i.e., the entire Pareto set.[4]

As we might expect, the situation where the uncovered set is maximal relative to the Pareto set is also the case where the yolk is maximal relative to the Pareto set. For the equilateral triangle, no point of the Pareto set is more than two yolk radii from the center of the yolk. Thus, the uncovered set is within two radii of the yolk, and is small relative to the yolk, but nonetheless includes the entire Pareto set.

Bounds for the Uncovered Set for the Case of Three Voters

McKelvey (1983) has shown that the uncovered set is always within 4 radii of the center of the yolk and our previous theorem is the stronger result that the uncovered set is always within 3.7 radii of the center of the yolk.

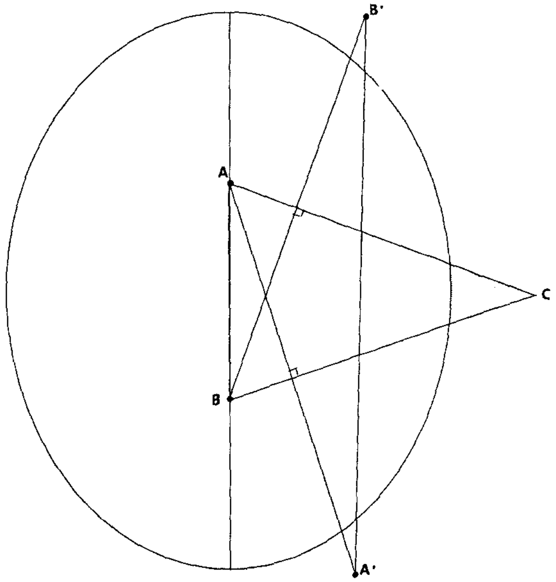

Fig. 5. The uncovered set is the entire triangle ABC (Pareto Set) except for the shaded region at C. The ellipse has foci A and B and major axis equal in length to line $A'B'$.

However, there are many cases where the uncovered set is much smaller than these upper bounds might suggest.

The previously discussed bounds are general upper (outer) and lower (inner) limits; for particular cases, with the number and location of voters specified, the actual size and shape of the uncovered set can often be specified precisely. In particular, for the case of three voters, the uncovered set is exactly described by a closed form analytic expression based on a geometric construction. Furthermore, preliminary calculations suggest that in the three voter situation, the uncovered set always lie within a circle centered at the center of the yolk with radius $2.83r$.

THEOREM 8. For the three voter case, the uncovered set can be determined by the ellipse-based construction method illustrated in Figure 5.

Proof sketch. The central idea is that, for each point, one can construct the win-set: any alternative in the Pareto set which is excluded from the win-set of the win-set of some point is automatically covered. The

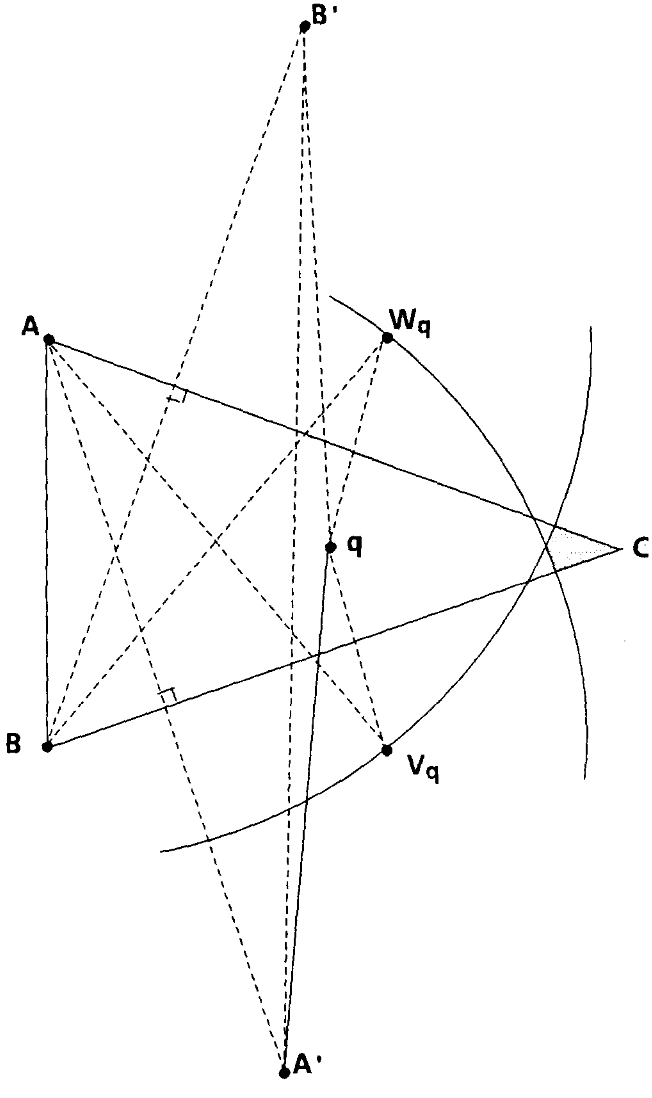

Fig. 6. Covering points near C with a general point q. The ellipse has foci A and B and major axis equal in length to line $A'B'$.

uncovered set than consists of all those points which are never excluded, regardless of which initial point was chosen.

The construction of the win-set of a win-set was illustrated in Figure 4, in which the initial point chosen was the center of the yolk. For the acute triangle shown, it is apparent that the limits of the win-set of the win-set near C are determined by the extreme points W and V of the "petals" (Figure 4), which, with the third petal oU, constitute the win-set of o. For obtuse triangles, the construction may be different (see Hartley and Kilgour 1987). The points near C, which lie closer to B than W or closer to A than V belong to the win-set of the win-set of o. This explains why the points shaded in Figure 4 are known to be covered.

Now suppose that the original point is moved from o to some nearby point q. Then W moves to W_q, the reflection of q in $\overline{AC}$, and V moves to V_q, the reflection of q in $\overline{BC}$. Let A' be the reflection of A in $\overline{BC}$, and B' the reflection of B in $\overline{AC}$. Then the distance from A to V_q is equal to the distance from A' to q, as shown in Figure 6.

As suggested by Figure 6, it is not difficult to show that all points near C which are covered by some point q are also covered by the point q' on $\overline{A'B'}$ nearest q. (Moving q perpendicularly toward $\overline{A'B'}$ in Figure 5 reduces both the distances $A'q$ and $B'q$, and therefore both radii $\overline{AV_q}$ and $\overline{BW_q}$). In other words, any point near C which is covered by some point q is covered by some point q' on the line $\overline{A'B'}$ (and within the triangle).

Now all points q on the line $\overline{A'B'}$ have the property that (the distances) $AV_q+BW_q=A'B'$, and it is not difficult to show that a point near C is covered if and only if the sum of its distances from A and from B exceeds $A'B'$. It is well-known that the points with distances from A and B summing to $A'B'$ form an ellipse with loci A and B and major axis $A'B'$. This justifies the construction in Figure 6. QED

This situation is somewhat different for some obtuse triangles (see Hartley and Kilgour, 1987 for details).

THEOREM 9. *For any triangle, a vertex (and nearby points) is included in the uncovered set if and only if the angle at that vertex is at least* $\pi/3$ $(=60°)$.

Proof sketch.[5] Let C be the vertex. Let θ be the angle made by $\overline{AC}$ and $\overline{BC}$. The construction which gave rise to the ellipse shown in Figure 5 is such that the ellipse which bounds the uncovered set or at vertex C has foci at A and B and can be shown to have its major axis of length 2:

$$z = \sqrt{AC^2 + BC^2 + 2AC \times BC \times \cos 3\theta}$$

When $\theta = \pi/3$, $\cos 3\theta = -1$ so that $z = BC + BC$. But this implies that C is on the bounding ellipse, i.e., it implies that A is uncovered. When $\theta < \pi/3$ then the expression for z shows that $z < AC + BC$, i.e., the ellipse falls inside the triangle near C. However, if $\theta > \pi/3$, 3θ becomes the exterior angle of triangle $A'B'C'$, and the construction fails – points near C are covered if and only if they are covered by C (see the proof of Theorem 8). QED

COROLLARY 1 to THEOREM 9. *For a triangle, the uncovered set includes the entire Pareto set if and only if the triangle is equilateral.*

Proof. Left as exercise for reader.

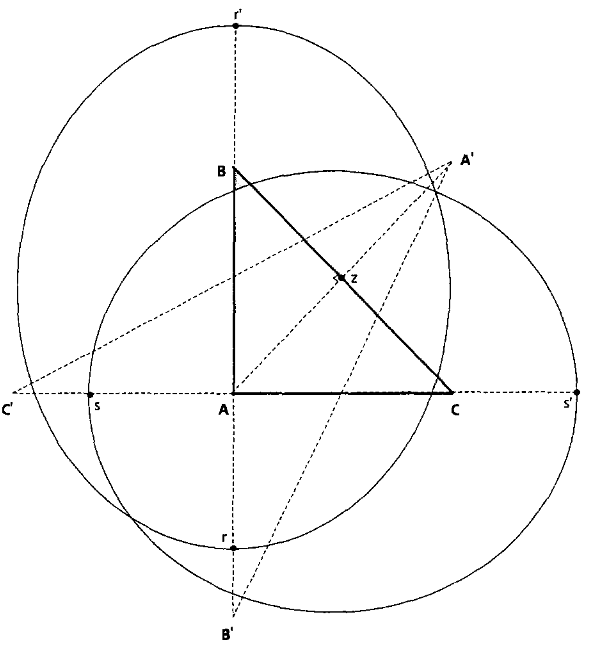

Fig. 7. Construction of uncovered set of isosceles right triangle. (Shaded points are covered).

In the equilateral triangle case, the distance from the center of the yolk to the most distant points on the edge (the vertices) is simply $2r$. Recall that is a special case of a result for regular polygons which we gave earlier.

Figure 7 shows the uncovered set of the isosceles right triangle example with vertices (0, 0), (0, 1), and (1, 0). Note that the uncovered set excludes points near both A and C, where the angles are $\pi/4$ ($=45°$). The ellipse with foci A and B has equation $5x^2+4y^2-4y=4$. The ellipse with foci B and C is analogous. One of the present authors conjectured at one time that the uncovered set is the set of points not covered by any point in the yolk. That conjecture is wrong. It can be shown that, in this example, not every covered alternative is covered by a point of the yolk; thus the yolk does not "generate" the uncovered set. In particular, the center of the yolk does not cover any points of the Pareto set in this example.

Figure 8 shows clearly the tendency of the uncovered set to exclude those alternatives in the Pareto set which are associated with an extreme voter, i.e., a voter with θ acute.

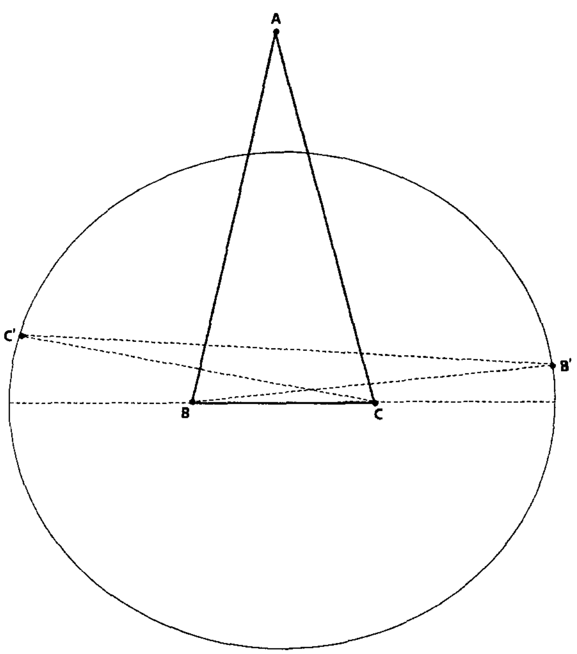

Fig. 8. Construction of the uncovered set in a typical case when two voter ideal points are relatively close. (Shaded points are covered).

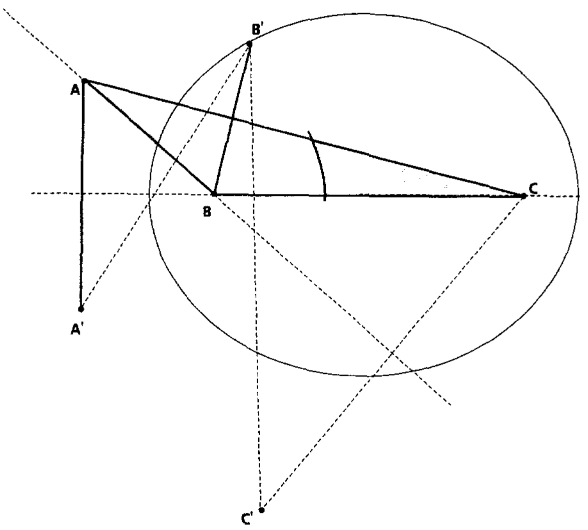

Fig. 9. Construction of a portion of the uncovered set for an obtuse triangle. (Shaded points are covered; construction of the other ellipse is similar).

In Figure 9, the uncovered set for an obtuse triangle is shown. Note that the ellipse with foci A and B has major axis not $A'B'$, but rather $AB + BB'$; see Hartley and Kilgour (1987) for the technical details. The ellipse with foci B and C is analogous. Note that only those alternatives near B are uncovered.

The uncovered set equals the Pareto set in the case of an equilateral triangle; in the opposite extreme, when the triangle contains an obtuse angle nearly 180°, only alternatives very close to that vertex are uncovered. In the limit, as the model becomes one-dimensional, only the median voter ideal point is uncovered.

V. THE BANKS SET

DEFINITION 13 (Banks, 1985). A Banks trajectory is a maximal chain, i.e., an ordered set of alternatives, a_1, a_2, a_3, ..., a_n, such that each alternative is majority preferred to all previous alternatives in the set, and such that there is no alternative outside the set that is majority preferred to every alternative in the set.

DEFINITION 14 (Banks, 1985). A *Banks point* is the last alternative in some Banks trajectory, i.e., a Banks point is the maximal element of a maximal chain, a_n.

DEFINITION 15 (Banks, 1985). The *Banks set* is the set of all Banks points.[6]

It has been shown that various agenda process, including the sophisticated outcomes of decisions under standard amendment procedure (Banks, 1985; Miller *et al.*, 1985), inevitably end up in the Banks set. Furthermore, it is easily shown that the Banks set is a subset of the uncovered set (Shepsle and Weingast, 1984; Banks, 1985). In finite cases, it has been shown that the Banks set can be often a proper subset of the uncovered set,[7] but there are no previous results on the relative sizes of the Banks set and the uncovered set in the spatial context.

Before we proceed to show the new result that, for the three voter case in the spatial context, the Banks set and the uncovered set are identical, some elaboration on the definition of the Banks set in the spatial context is useful. In particular, we shall distinguish between continuous and discontinuous Banks trajectories.

DEFINITION 16. In the spatial context, a *continuous Banks trajectory* is one in which the elements of the trajectory form a continuous path in the space (i.e., one never needs to lift pencil from paper in tracing the path).

DEFINITION 17. In the spatial context, *a discontinuous Banks trajectory* is one in which the elements of the trajectory do not form a continuous path in the space.[8]

We show in Figure 10 a three voter example based on a simulation of a finite grid of discontinuous Banks trajectories, using the right triangle with vertices (0, 10), (10, 0) and (0, 0). A is the first point in the trajectory. Points labeled 1 are in the inverse win-set of A. B is the second point in the trajectory. Points labeled 2 are those in the inverse win-set of B, which have not previously been excluded because they are also in the win-set of A etc. Because these are Banks trajectories, each point in the

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111
1111
11111
111111
1111111
11111111
111111111
1111222111
12223▒3B22A
122233333322
1222223333332
11222333333332
111233333334442
1111113333446521
11111543344754421
111115553465444211
11111G55F4544442111
1111666E5D4444432111
111114▒444344443▒21111
11114444444444333211111
111114444444333321111111
11111444C333333211111111
111113333333333211111111
1111113333333321111111111
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Fig. 10. A Noncontinuous Banks Trajectory beginning outside the Pareto Set for a three voter majority rule game. (Voter ideal points are shown shaded).

sequence ... *CBA* is majority preferred to all subsequent points in the sequence. In this example the trajectory has seven elements, ending at *G*. Note that the regions which are simultaneously excluded as new points are added to the trajectory may be widely separated.

THEOREM 10. *For the three voter case in the spatial context, the Banks set and the uncovered set are identical, provided that discontinuous Banks trajectories are allowed.*

Proof sketch. Since the Banks set is a (not necessarily proper) subset of the uncovered set, all that must be shown is that any point in the uncovered set is the maximum element of a Banks trajectory.

Pick any point x in the Pareto set. Draw the win-set of x which, in general, will have three petals. (See e.g. Figure 4). Choose the widest petal. In Figure 4, this is $\overline{oU}$. Let R and S be the points of intersection of this widest petal with the corresponding side of the triangle (extended, if necessary). Begin with the paths Rx and Sx, along the boundaries of the petal. Displace each path outward infinitesimally, holding the endpoint at x fixed, resulting in $R'x$ and $S'x$, as shown in Figure 11.

The required trajectory alternates points from $R'x$ with points from $S'x$

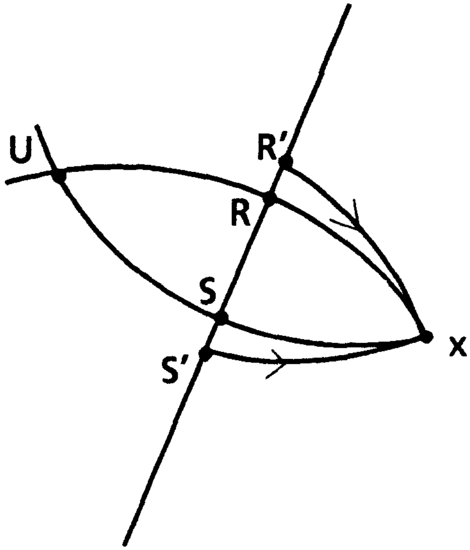

Fig. 11. Construction to define a Banks Trajectory with x as its maximal element for x in the uncovered set.

as it steadily moves toward x, the top element. It can be shown that there are points (located in the petal xU) which beat x *and* every point of $R'x$ *and* every point of $S'x$ if and only if x is covered. If follows that, if x is uncovered, this zigzag path is a Banks trajectory with x as its maximal element. Hence, for any element in the uncovered set we can find a discontinuous chain of which it will be the maximal element, i.e., every uncovered point is also a Banks point. QED

Since we know how to specify the location of the uncovered set in the three voter case, we now know the exact location of the Banks set for the three voter case. We conjecture that, in the spatial context, the Banks set and the uncovered set will always be identical, but we have not yet been able to prove this result.

The Banks set has another important property. For a two dimensional issue space, the Banks set specified the domain within which the outcomes of the manipulation of committee jurisdictional assignments must be confined. To prove this, we state a useful theorem about the Schatt-schneider set and then show that the two-dimensional Schattschneider set, which we have shown to be a subset of the uncovered set, is also a subset of the Banks set.

THEOREM 11. *In some multidimensional issue space, if committee jurisdictions are what Krehbiel (1984) refers to as a simple institutional arrangement, i.e., are one-to-one match-ups between orthogonal issue dimensions and committee jurisdictions, then the locus of possible outcomes of the open agenda process with a germaneness rule (under either sincere or sophisticated voting) is given by the Schattschneider set (see Figure 1 above).*

Proof. See Grofman and Feld (1986).

In other words, if committee jurisdictions (i.e., a set of orthogonal axes) are subject to manipulation, then the Schattschneider set specifies the domain within which the outcomes of any such manipulations must be found.

THEOREM 12. *The Schattschneider set is a subset of the Banks set.*

Proof sketch. The Schattschneider set consists of median points on median lines. For any such line we can construct a Banks trajectory on the line which has the median voter projection as its maximal element. To do so, all we need do is move nearer and nearer the median voter projection, in a zigzag trajectory from one side of the median to the other. Futhermore, there is no point of the line which can defeat *all* points on the line. If there were, that point would be majority winner, but if it were a majority winner it would necessarily be the median point on all median lines. Thus every Schattschneider point is a maximal element of a maximal chain and thus it is also a Banks point. QED

The converse of Theorem 12 is not true. There are many Banks points which are not also Schattschneider points.

VI. CONCLUSIONS

We have specified new and tighter general bounds on the uncovered set in the spatial context, and in particular, we have shown how to find the exact boundaries of the uncovered set in the three voter case. In general, we have shown that the uncovered set is small relative to the Pareto set when the yolk is small. This is important because it is known (Cox, 1985; Feld *et al.*, 1985) that the yolk tends to be small if there is a degree of bilateral symmetry in the array of voter ideal points. However, we have also shown that the uncovered set can be the entire Pareto set in certain extreme cases. Thus, it is necessary to analyze the particular situation to determine whether one can be sure that voting outcomes can be expected to be confined to a small portion of the feasible set. Further research needs to be done, however, to precisely locate the uncovered set in specific cases with more than three voters, although we have provided an exact result for the case of regular polygons.

We believe that subsets of the uncovered set like the Schattschneider set and the Banks set will arise from specific institutional features of group decision processes, and that further work must be done to identify other such subsets and the institutional arrangements which give rise to them. Nonetheless, by providing new results on the uncovered set in the spatial context, this paper contributes to the continuing study of the institutional and contextual features of majority rule which bring order and stability into collective decision making. It is largely these "fine-grained" features of the structure of majority preference in the spatial context that make politics as we know it possible and prevent a disintegration of majority rule processes into indeterminacy and instability (Riker, 1982).

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NOTES

[1] If there are more than two dimensions, we would, of course, need to replace circle with sphere, line with hyperplane, etc., in the definitions and results given below. Also, results for the Schattschneider set do not straightforwardly generalize. In particular, in the n-dimensional case, the Schattschneider set need not be in the uncovered set, although in the 2-dimensional case, it must be. (See Grofman and Feld, 1986).

[2] Let q be outside the win-set of the win-set of r. Clearly, r beats q. If there were an s that q beats and that r does not beat, then q beats s and s beats r, so that q would be in the win-set of s, which would be in the win-set of r, which contradicts our assumption that q is outside the win-set of the win-set of r. Thus, any point which is not in the win-set of some point in the win-set of r is covered by r.

[3] A direct implication of Theorem 6 is that the minimum win-set of a point, x, outside of the yolk, is contained within the minimum win-set of a point y that is further from the center of the yolk in the same direction; and the maximum win-set of a point x is contained within the maximum win-set of a point y that is further from the center of the yolk in the same direction as x. In other words, for points outside the yolk, as we move further off in a given direction, the set of points that must be beaten and the set of points that may be beaten both shrink in size. These results do not hold true inside the yolk. Although a point $0.4r$ from the center of the yolk covers all points directly opposite it on the other side of the yolk at a distance $3.7r$ or more from the center of the yolk, points *within* the yolk and less than $0.4r$ from the center may not cover all of these same points.

[4] The basic idea of the proof was independently arrived at by two different subsets of the present authors. A complete version of the proof is found in Hartley and Kilgour (1987).

For more than three voters, it is possible to create situations that are arbitrarily close to the three voter situation by putting the voter ideal points in three groupings arranged in a pattern close to that of an equilateral triangle. In that case, the uncovered set would be essentially be the entire Pareto set.

[5] The basic idea of this proof was independently arrived at by two different subsets of the present authors. An alternative derivation of the proof is found in Hartley and Kilgour (1987).

[6] Banks (1985) does not call the set whose properties he was the first to identify the Banks set. Following Miller *et al.* (1985) we give it that label, which seems to us the most appropriate.

[7] In particular, when there are fewer than 7 alternatives, the Banks set and the uncovered set must be identical. However, for a larger number of alternatives, the Banks set can be much smaller than the uncovered set. (Miller *et al.*, 1985).

[8] It is not obvious that continuous and discontinuous Banks trajectories will have the same sets of maximal elements. Moreover, the maxima of some continuous Banks trajectories may not even be Banks points, since the top of a continuous trajectory may be a local but not a global maximum. Nonetheless, continuous trajectories are of importance because they can be thought of as the limiting outcomes of "incremental" decision making – a sequence of tiny adjustments to the status quo.

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