THE SPATIAL MODEL OF SOCIAL CHOICE AND VOTING

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Abstract

This chapter presents the basic elements of the standard spatial model commonly used as a framework for developing theories of legislative, electoral, and other forms of social choice and voting and that is increasingly used in empirical analysis as well. It introduces the concepts of single-peaked and Euclidean preferences, win sets, the core, the Condorcet winner, median lines, the yolk, and the uncovered set, and presents such foundational results as Black's Median Voter Theorem, Plott's Majority Rule Equilibrium Theorem, McKelvey's Global Cycling Theorem, and Greenberg's Core Existence Theorem.

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4.1 *THE SPATIAL MODEL OF SOCIAL CHOICE AND VOTING*

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This chapter presents the basic elements of the standard spatial model commonly used as a framework for developing theories of legislative, electoral, and other forms of social choice and voting and increasingly used in empirical analyses as well. It builds on Chapters 3.1 and 3.2 by placing concepts introduced there in a spatial context, and it lays out some groundwork for the remaining chapters in this section and the first two chapters in Section 6.

1. Overview

'The spatial model' is often associated with the work of Anthony Downs (1957, Chapter 8), who drew on some remarks by Harold Hotelling (1929) to propose that two candidates or parties competing for the support of an electoral majority (as discussed in the Chapter 4.2) would 'converge to the center'. However, the origins of the formal spatial model of social choice lie in Duncan Black's (1948, 1958) attempt to build a 'pure science of politics' founded on 'a point-set representing motions' to be voted on by a 'committee' (i.e., a small set of voters). In effect, Black (and Downs more explicitly) formalized the notion of the left-right political spectrum that originated with the seating arrangements in the National Assembly at the time of the French Revolution. Formal political theorists have taken this classic notion and run (very) far with it. Driven by desire for generality, explicitness of assumptions, and theoretical coherence, they have produced an elaborate theory with possibly intimidating terminology, notation, concepts and derivations. Here we attempt to simplify or sidestep most of these complexities but still convey the main ideas. We follow the style of formal theorizing by presenting basic conclusions as numbered propositions, so that we can conveniently refer back to them.

The essential idea underlying Black's 'point-set representing motions' is that a geometrical space (a one-dimensional line, a two-dimensional plane, a three-dimensional solid, etc.) can represent the 'space' of (potential) 'policy alternatives' or 'political outcomes' available for social choice, whether on some narrowly defined issue or in a more global sense. Given voters with plausible preferences over the space and a voting rule (and perhaps more elaborate political institutions), we can derive logical results pertaining to the existence and nature of expected outcomes and general relationships among preferences, institutions and outcomes.

The spatial model captures our sense that there may be many (in principle, an infinite number of) policy alternatives available for choice and that these alternatives may be related in a 'spatial' sense — for example, some alternatives are 'close together' while others are 'far apart' and compromise alternatives always exist 'between' policy alternatives; furthermore, some alternatives may be described as 'left-wing' (or extreme in some other sense), others as 'right-wing' (or extreme in an opposite sense), and still others are 'centrist' (or otherwise moderate), with essentially infinite gradations in between. It further captures our sense that voter preferences are likely to be structured spatially as well — for example, that some voters are 'close together' with mostly similar preferences over alternatives while other are 'far apart,' e.g., 'left-wingers' and 'right-wingers' with largely opposed preferences, and still others are 'centrists,' again with gradations in between.[1]

A common theme concerning the spatial model is that in a one-dimensional setting social choice — and majority rule in particular — is well behaved but in a higher-dimensional setting its operation becomes 'chaotic.' This chapter will both explain in what sense this is true and suggest why the 'chaotic' characterization may be overdrawn.

2. Voter Preferences

As discussed in Chapters 3.1 and 5.1, social choice theory assumes that individuals have preference orderings over all alternatives available for choice; this means that individual preference is assumed to be *transitive*, i.e., a voter who prefers x to y and y to z also prefers x to z. In the general case, no restrictions are put on individual preference orderings (thus the 'Universal Domain' condition discussed in Chapters 5.1 and 5.2) but, given a spatial context, it is natural to make particular assumptions concerning voter preferences.

2.1 Single-Peaked Preferences

Consider a one-dimensional alternative space, i.e., a single ideological dimension or a single issue concerning which alternatives differ in only one respect. Any point along the line represents an alternative potentially available for social choice — indeed, we will often use the term 'point' in place of 'alternative' — and voters have preferences over these alternatives. It is natural to assume that each voter has a point of maximum preference called his *ideal* (or *bliss*) *point* (and, in that sense, has a spatial location), and that the voter's preferences between pairs of other alternatives relate to their distance from this ideal point. In the case of a one-dimensional alternative space, a standard assumption is that voters have what Black (1948, 1958) called *single-peaked* preferences: given two alternatives x and y that lie on the same side of his ideal point, a voter prefers the closer one (whereas, if x and y lie on opposite sides of his ideal point, he may prefer x to y, or prefer y to x, or be indifferent between them). Such preferences are called 'single-peaked' because a graph in which the horizontal axis represents the alternative space and the vertical axis represents a voter's degree of preference (or 'utility') for alternatives is literally single-peaked, rising steadily and reaching a (single) peak at the voter's ideal point and declining steadily thereafter. However, single-peaked preferences may be asymmetric about the peak. Accordingly, two voters with identical ideal points may have different preferences with respect to other alternatives, which means that (i), given only the location of his ideal point, we cannot infer a voter's preference between two alternatives lying on opposite side of that ideal point and (ii), given only that a voter prefers x to y, the only thing that we can infer about the location of his ideal point is that it lies on x side of y.

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[1] In discussing the one-dimensional spatial model, we use left-right terminology, but other (non-ideological) language may be appropriate in particular situations (e.g., low- vs. high-spending proposals). Of course, the spatial model allows for multiple independent ideological dimensions, e.g., left-right with respect to economic issues, left-right with respect to social issues, etc.

The set of points that a voter prefers to x is called his *preferred set* with respect to x. Visualizing preferred sets in terms of a single-peaked graph leads to the following conclusions.

Proposition 1. Given single-peaked preferences,

(a) a voter's preferred set with respect to x is a line segment extending from x through his ideal point to a point x' such that he is indifferent between x and x';
(b) if a voter prefers y to x, his preferred set with respect to y is a subset of his preferred set with respect to x;
(c) the preferred sets with to respect x of two voters are either disjoint (if they have ideal points on opposite sides of x) or one is a subset of the other (if they have ideal points on the same side of x).

Point (c) implies that the preferred sets with respect to x of all voters with ideal points on the same side of x are ordered in terms of their inclusiveness.

Single-peakedness can be generalized to two or more dimensions by assuming that voter preferences over the points on every straight line through the space are single-peaked.[2] In two dimensions, this implies that voter *indifference curves*, i.e., sets of points among which a voter is indifferent, are closed curves concentric about the voter's ideal point that everywhere bend inward and that a voter's preferred set with respect to x is the set of points enclosed by his indifference curve that passes through x.

2.2 Euclidean Preferences

Another assumption about spatial preferences — consistent with single peakedness but stronger — is that voters have *Euclidean* preferences. This means that between *any* two alternatives, a voter prefers the one that is closer to his ideal point and is indifferent between equidistant alternatives so, in the one-dimensional case, the single-peaked graph is *symmetric* about a voter's ideal point.

Given this assumption, each voter's preferences are fully specified once we know the location of the voter's ideal point (and voters with the same ideal point have identical preferences over all alternatives). In particular, we know a voter preference between x and y even if they lie on opposite sides of his ideal point and, if i prefers x to y, we can infer that his ideal point is closer to x than y. We can strengthen Proposition 1 as follows.

Proposition 2. Given Euclidean preferences,

(a) a voter's preferred set with respect to x is a line segment extending from x through his ideal point to a point x' an equal distance beyond;
(b) the preferred set with respect to x of one voter is a subset of that of another voter only if the ideal point of the former is closer to x than that of the latter.

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[2] In the terminology of social choice theory, such preferences are said to be *compact*, *continuous*, and *strictly quasi-concave*.

With respect to (a), since x and x' are equidistant from the voter's ideal point, it is natural and convenient to refer to x' as the *reflection* of x through the ideal point. Point (b) implies that the ordering of preferred sets of voters with ideal points on the same side of x in terms of their inclusiveness is the same as the ordering of their ideal points in terms of distance from x.

The definition of Euclidean preferences applies directly to higher dimensions. In two (or more) dimensions, this implies that a voter's indifference curves are concentric circles (or spheres, etc.) about his ideal point.

2.3 Social Preference

The following analysis focuses on the *social preference relation* over alternatives, given some number n of voters with single-peaked or Euclidean preferences and some *proper k-majority rule*, i.e., where k is an integer greater than $n/2$ — that is, *simple majority rule* (i.e., when k is the smallest integer greater than $n/2$) or some more 'demanding' *supermajority rule* (i.e., with some larger k, as discussed in Chapter 3.2). Alternative x is *socially preferred* to y if at least k voters out of n prefer x to y.[3] Since this is an awkward phrase, we shall usually say that 'x beats y.'[4] Such social preference relations, especially simple majority rule, govern various many political processes, e.g., committee decisions, legislative voting, and mass elections, analyzed using the spatial model.

The set of alternatives that beat x under k-majority rule is called the *win set* $W(x)$ of x. If $W(x)$ is empty, x is *unbeaten*. The *core* is the set of all unbeaten points. Because social preference may *cycle* (if k is less than n), as discussed in Chapter 3.1 with respect to simple majority rule, the core may be empty. Keeping voter preferences are constant, it is clear that, as the voting rule becomes more demanding, win sets contract and the core expands; as the voting rule becomes less demanding, win sets expand and the core contracts and may disappear. An empty core implies *instability*, in that for every outcome x there is a set of voters with both the power and desire to replace x with some other alternative; a large implies *non-responsiveness* as voter preferences change, which may lead to conflictful behavior and 'gridlock.' From this perspective, the optimal situation is the existence of a small core, ideally a single alternative.

Alternative x *Pareto-dominates* y if at least individual prefers x to y and none prefers y to x. The *Pareto set* consists of all alternatives that are not Pareto-dominated and is equivalent to the core under (relative) unanimity rule.

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[3] This defines the *absolute* variant of k-majority rule, which also has a *relative* variant in which n is replaced by the (perhaps smaller) number of voters who are not indifferent between x and y. While indifference does occur in a spatial context (and is critical in defining the boundaries of sets), it is sufficiently rare that definitions and results are essentially the same under both variants.

[4] With some abuse of language, we may also say that 'x ties y' (or 'y ties x') if x does not beat y and y does not beat x. Under simple majority rule with an even number of (non-indifferent) voters, 'x ties y' means x and y are tied in the ordinary sense, but in the general case it means only that neither x nor y is preferred over the other by the requisite k voters.

3. The One-Dimensional Spatial Model

We suppose that there are n voters with ideal points $x^1, x^2, \ldots, x^n$, numbered according to their spatial locations with x^1 the left-most and x^n is the right-most. (If some ideal points coincide, their numbering is arbitrary.)

Proposition 3. Given a one-dimensional space, n voters with single-peaked preferences, and any proper k-majority rule,

(a) every non-empty win set of x is identical to some voter's preferred set with respect to x, so
(b) $W(x)$ is a line segment with one end at x; moreover
(c) if x is beaten by any point on one side of x, x beats every point on the other side of x;
(d) if x beats y, $W(x)$ is a subset of $W(y)$; and
(e) if x beats y and y beats z, then x beats z, i.e., social preference is transitive.

By definition, x is beaten by y if and only if y lies in (the intersection of) the preferred sets of at least k voters; given single-peaked preferences, the ideal points of these voters all lie on the y side of x. Thus, $W(x)$ is either empty (if fewer than k ideal points lie on either side of x) or $W(x)$ is equal to the k^{th} most inclusive individually preferred set with respect to x (if at least k ideal points lie on the same side of x), in which case it inherits the property of individual preferred sets given by Proposition 1(a). Moreover, (c) in this event at least k voters prefer x to every point on the other side of x. With respect to (d), y belongs to $W(x)$ so, if x is unbeaten, (d) follows trivially; if $W(x)$ is not empty but lies on the y side of x, (d) follows immediately because y does not belong to $W(x)$; if $W(x)$ lies on the side of x opposite y, by Proposition 1(b) the k or more voters with ideal points on that side of x all have more inclusive preferred sets with respect to y than to x. Point (e) follows directly from (d) because set inclusion is transitive. This means that the cyclical social preference phenomenon cannot arise with single-peaked preferences.

In the one-dimensional case, k-majority rule creates two *pivotal* voters — the *right pivot* whose ideal point x^k is $n-k$ positions in from the ideal point of the right-most voter and the *left pivot* whose ideal point $x^{k'}$ (where $k' = n - k + 1$) is $n-k$ positions in from the ideal point of the most left-wing voter.

The absence of cyclical social preference given single-peaked preferences implies that the core is never empty. The following proposition specifies its precise extent.

Proposition 4. Given a one-dimensional space, n voters with single-peaked preferences, and any proper k-majority rule, the core is the interval between (and including) the ideal points of the two pivots.

Let y designate any point between and including the pivotal ideal points and let z designate any other point. Necessity follows because (at least) the right pivot and the n-k voters to his left prefer y to any point to the right of y, and (at least) the left pivot and the n-k voters to his right prefer y to any point to the left of y, so in any event (at most) $k-1$ voters prefer any other point to y.

Sufficiency follows because (at least) the closer pivot and the k voters in the direction of the other pivot prefer the closer pivotal ideal point to z.[5]

Under unanimity rule ($k = n$), the entire *Pareto set*, extending from x^1 to x^n, is unbeaten, and the core contracts as the decision rule becomes less demanding. At the limit, if the number of voters n is odd and simple majority rule is used, so $k = k' = (n+1)/2$, the *median voter* (with ideal point x^{med}, where $med = (n+1)/2)$) is both the left and right pivot. (Possibly two or more voters share this median ideal point.) The most famous result pertaining to the spatial model follows as a corollary of Proposition 4.

Corollary 4.1 (Black's [1948, 1958] Median Voter Theorem). Given an odd number of voters with single-peaked preferences and simple majority rule, x^{med} is the unique unbeaten point; moreover x^{med} is a Condorcet winner.

As discussed in Chapter 3.1, a *Condorcet winner* is an alternative that beats every other alternative under simple majority rule. Thus, in the absence of agenda control, x^{med} is the expected voting outcome. Whatever the status quo, some voter (indeed, a majority of them) has an incentive to propose x^{med}, which can then beat the status quo and any rival proposals. This property of x^{med} drives the 'convergence towards the center' (i.e., the median ideal point) associated with Downs (1957). It also means that in legislative or electoral models of majority rule voting with single-peaked preference, the median voter can, with respect to determining the winning outcome, 'stand in' for the set of all voters.

If simple majority rule is used but the number of voters n is even, $k = n/2 + 1$ and $k' = n/2$, so there is a (small) interval of unbeaten points between $x^{n/2}$ and $x^{n/2+1}$ but there is no Condorcet winner (unless these two ideal points happen to coincide).

Given Euclidean preferences, Proposition 2(b) implies that Proposition 4 can be strengthened as follows.

Proposition 5. Given a one-dimensional space, n voters with Euclidean preferences, and any proper k-majority rule, x beats y if and only if both pivots prefer x to y.

If x and y lie on the same side of both pivots, this proposition adds nothing to Proposition 4 but, if x and y lie on opposite sides of the pivots, Proposition 5 is stronger and has this corollary.[6]

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5 An unbeaten point has a *tie set* that extends in both directions through the ideal points of the two pivots. A beaten point has a win set that extends through the ideal point of the closer pivot and a tie sets that lies beyond the win set and extends through the ideal point of the more distant pivot.

6 Given Euclidean preferences, an unbeaten point has a tie set that extends in both directions to its reflection through the ideal points of the pivots, and a beaten point has a win set that extends to its reflection through the ideal point of the closer pivot and a tie set that lies beyond its win set and extends to its reflection through the ideal point of the more distant pivot.

Corollary 5.1. Given an odd number of voters with Euclidean preferences, and simple majority rule, x beats y if and only the median voter prefers x to y, i.e., if x is closer to x^{med} than is y.

Thus all win sets are identical to the median voter's preferred sets (and $W(x)$ extends from x to its reflection through x^{med}). The other $n - 1$ voters in effect disappear from view, and the median voter can in every respect 'stand in' for the set of all voters, in the sense that changes in location of, or the addition or deletion of, other voters that do not change the location of the median voter have no effect on social preference.

In sum, given single-peaked preferences over a one-dimension space, social choice operates in an orderly fashion and unbeaten alternatives always exist. If an alternative x is beaten, it is beaten only by alternative on one side of x, reflecting the preferences of a single set of voters of sufficient size given the voting rule. Given simple majority rule with an odd number of voters, social preference mimics individual preference; indeed, if preferences are Euclidean, social preference is identical to that of the median voter.

4. The Two-Dimensional Spatial Model

We now consider a two-dimensional alternative space — that is, any point on a plane represents a possible alternative and n voter ideal points are distributed over the plane. For the most part, we will restrict our attention to the case of Euclidean preferences and, in contrast to the one-dimensional case, we will first consider the case of simple majority rule with an odd number of voters. In the final section of the chapter, we shall consider what happens as k-majority rule becomes more demanding.

4.1. Win Sets in Two Dimensions

In the one-dimensional case, Proposition 1(c) tells us that the preferred sets of two voters with respect a given point x can be related in only two ways: they are disjoint or one is a subset of the other. This implies that x can be beaten through the preferences of only a single majority (or larger set) of voters. In contrast, in two (or more) dimensions, the preferred sets of two voters with respect a given point x almost always overlap without one being a subset of the other. This implies that, in contrast to Proposition 3(a), a win set is almost never identical to the preferred set of any individual voter; rather it is an amalgamation of the preferred sets of many (typically all) voters, reflecting the fact that different majorities prefer different sets of alternatives to x. This phenomenon has important consequences for social choice.

Figure 1, which shows the ideal points of five voters, illustrates this phenomenon. Note that the Pareto set is the *convex hull* of the ideal points, i.e., the set of points enclosed by the polygon (a triangle in this case) with vertices at the 'non-interior' ideal points. Figure1 shows voter preferred sets with respect to point x as shaded areas that become darker they intersect. Given simple majority rule ($k = 3$), the win set of x includes all points that lie in the intersection of three preferred sets. The boundary of $W(x)$ is shown by dark line and it has four distinct 'petals' (two of which overlap to form a single 'leaf') pointing in different directions, reflecting the preferences of four distinct majorities: voters 1, 2, and 4 with respect to the right-pointing petal; voters 1, 3, and 5 with respect to the left-pointing petal; voters 2, 3, and 4 with respect to the left downward-pointing petal; and voters 2, 3, and 5 with respect to the right downward-pointing petal.[7]

Note that point x in Figure 1 is beaten by points that lie 'on all sides' of x; more specifically, $W(x)$ is *disorderly* in that it is broken up into (non-overlapping) 'petals' that do not lie entirely to one side of a line through x. In contrast, $W(x)$ is *orderly* if it is composed entirely of overlapping 'petals' that form a single 'leaf' that lies entirely to one side of such a line in the manner of an individual preference set — indeed, in the neighborhood of x it corresponds to the preferred set of a single voter.

Consider any straight line L through the space. Each voter has an *induced ideal point* on L, i.e., his most preferred of all points on L (given Euclidean preferences, the point on L closest to his ideal point, i.e., the projection of the ideal point on L), one of which is the median induced ideal point. Thus, with respect to majority preference over the points on L, all propositions for the one-dimensional case apply to any number of dimensions, giving us the following.

Proposition 6. Given a space of any number of dimension, n voters with generalized single-peaked preferences, and any k-majority rule, the win set of every point x is

(a) *starlike*, i.e., if x is beaten by y, x is beaten by every point on the straight line segment between y and x; and
(b) *semi-polarized*, i.e., if x is beaten by y, x beats every point on the straight line through x and y that lies on opposite side of x from y.

Point (b) establishes the fact that, while in two (or more) dimensions different majorities (or larger sets of voters) may pull in *different* directions, they never pull in *directly* opposite directions. For the special case of simple majority rule with an odd number of voters, (b) can be strengthened as follows.

Proposition 6′. Given a space of any number of dimension, an odd number of voters with generalized single-peaked preferences, and simple majority rule, the win set of every point x is *almost polarized*, i.e., x is beaten by points on (one side of) almost every line L through x.

However, x beats every point on at least one line, and perhaps some larger (but finite) number of such lines, on which x is the median induced ideal point. One such line is tangent to an orderly win set; multiple such lines separate the non-overlapping petals of disorderly win sets. Under more demanding rules, in contrast, x is unbeaten by any points on a substantial proportion of all lines through x.

4.2 Median Lines and Condorcet Winners

There is one circumstance in which preferred sets of two voters with respect to x either are disjoint or one contains the other (as in the one-dimensional case); this is when their ideal points lie on a straight line that passes through x, so their preferred sets are tangent to each other. If x lies

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[7] The intersection of the latter two petals constitutes the single petal of the win set of x under super-majority rule with $k = 4$. (Supermajority win sets can also have multiple petals.) The win set of x under unanimity rule is empty, because x lies in the Pareto set.

between the ideal points, the preferred sets are disjoint; if the ideal point of one voter lies between x and the ideal point of the other voter, the preferred set of the later contains the preferred set of the former. This hints at how a Condorcet winner can exist in two-dimensions.

Any straight line L partitions the set of voter ideal points into three subsets: those that lie on one side of L, those that lie on the other side of L, and those that lie on L itself. A *median line* (or *bisector*) partitions the set of ideal points so that no more than half lie on either side. Given that n is odd, there is exactly one median line perpendicular to any line L through the space, and their intersection is the median induced ideal point on L and beats every other point on L. Every ideal point lies on at least one median and every median line passes through some ideal point. While almost all median lines pass through just one ideal point, a finite number of *limiting* median lines pass through two (or possibly more) ideal points. Typically pairs of limiting median lines pass through a given ideal point, with non-limiting median lines sandwiched them. Figure 2 shows all limiting median lines for the configuration of ideal points in Figure 1. The shaded cone indicates all the (non-limiting) median lines that pass through x^1.

Consider any two points x and y lying on a line L in a two-dimensional space. The preferences between x and y of voters with Euclidean preferences depend on which side of the perpendicular bisector of the line segment between x and y their ideal points lie . If the median line perpendicular to L lies on the x side of this bisector, a majority of ideal points lie on that side of the bisector, so x beats y, and if it lies on the y side, y beats x. As noted above, if x is the point of intersection with the median line and L, x beats all other points on L, giving us the following proposition.

Proposition 7. Given a two-dimensional space, an odd number of voters with Euclidean preferences, and simple majority rule, every point that lies off some median line is beaten by a point on that median line.

Generalizing to all median lines gives us this necessary and sufficient condition for a Condorcet winner in a two-dimensional space.

Proposition 8. Given a two-dimensional space, an odd number of voters with Euclidean preferences, and simple majority rule,

(a) a point x is unbeaten if and only if every median line passes through x; in which case
(b) x is a Condorcet winner; and
(c) x is the ideal point of some voter.

Point (c) follows because an infinite number of median lines pass through x while there are only a finite of ideal points, so only a finite number of median lines can pass though ideal points other than x. Since (with n odd) every median mine must pass through an ideal point, it must be that x is an ideal point.

Provided that no ideal points coincide, the configuration of ideal points must meet the following *Plott symmetry* condition in order to meet the condition specified in Proposition 8.

Corollary 8.1 (Plott's [1967] Majority Rule Equilibrium Theorem). Given a two-dimensional space, an odd number of voters with Euclidean preferences, and simple majority rule, a point x is unbeaten (and a Condorcet winner) if one ideal point is located at x and the other $n-1$ ideal points can be paired off in such a way that the two points in each pair lie on a straight line with, and on opposite sides of, x.[8]

Given Plott symmetry, the voter with ideal point at x is the two-dimensional counterpart of the one-dimensional median voter, giving us the two-dimensional counterpart of Corollary 5.1.

Corollary 8.2. (Davis, Degroot, and Hinich's [1972] Social Preference Ordering Theorem). If and only Plott symmetry holds, majority preference is identical to the preference of the voter with ideal point at the central point x, i.e., y beats z if and only if y is closer to x than is z, and is therefore transitive.

But the Plott symmetry condition is very stringent, making it extremely unlikely that a Condorcet winner exists in two-dimensional space, so we need to consider what happens in its absence.

4.3 The Yolk and Global Cycling

Given a finite set of alternatives with unrestricted preferences (e.g., in the tournament setup discussed in Chapter 3.1), the absence of a Condorcet winner under majority rule leads us to look to the *top cycle set*, i.e., the minimal set of points such that every point in the set beats every point outside the set, to locate the expected voting outcome.[9] However, in two well-known papers, McKelvey (1976 and 1979) demonstrated that, for Euclidean and much more general preferences respectively and in the almost certain event that Plott symmetry does not hold, the top cycle set encompasses the entire space. From this result, McKelvey (1976) drew some implications for voting that have to some extent bedeviled voting theorists ever since. He observed that, if an agenda setter knows the preferences of all voters who always vote accordingly, he would have total control over the voting process. More specifically, he could design an sequence of votes leading from *any* status quo point to *any other* point as the voting outcome, most plausibly to his own ideal point but even to a point well outside the Pareto set. More generally, others have concluded that, in the face of such global cycling, 'anything can happen.'

Taken together, the Plott and McKelvey theorems seem to suggest a puzzling discontinuity in the two-dimensional spatial model: given Plott symmetry, majority rule is wholly coherent but, given the slightest perturbation of ideal points, the character of majority rule changes entirely and becomes totally chaotic. We now introduce a concept that both mitigates this puzzle and provides the basis for an intuitive understanding of McKelvey's theorem and its implications.

While it is very unlikely that ideal points are distributed so that all median lines have a common intersection, it is likely that they typically pass through a fairly small central region of the

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[8] Plott (1967) stated his theorem for generalized single-peaked preferences, in which case the condition requires that voters can be paired off in such a way that their 'contract curves' all pass through x. (This result actually was anticipated by Black and Newing, 1951). Enelow and Hinich (1983) show that this condition is not quite necessary: if two or more voters have ideal points at x, the symmetry requirement is weakened, as all that is needed is that fewer than half of the ideal points lie to one side of any line through x. Subsequent references to Plott symmetry incorporate this generalization.

[9] In a tournament setup, this set contains a complete cycle, hence its name.

space. Following an early insight by Tullock (1967) later formalized by Ferejohn, McKelvey, and Packel (1984) and McKelvey (1986), we define the *yolk* as the region bounded by the circle of minimum radius that intersects every median line. The yolk is described by its *center* c, which indicates the location of the generalized center (in the sense of the median) of the distribution of ideal points, and its *radius* r, which indicates the extent to which the configuration of ideal points departs from Plott symmetry. Figure 2 shows the yolk for the five-voter configuration in Figure 1, as well as the *yolk triangle*, i.e., the three 'outmost' median lines that intersect the yolk only tangentially.

Given only the location and size of the yolk, let us consider what points may beat a given point. Figure 3 shows a point x at distance d from the center of the yolk (where $d > r$, so x lies outside the yolk) and a line L through x. Of course, if we knew the location of all ideal points, we could locate the median line perpendicular to L and thereby identify the precise set of points on L that beat x — namely, all the points between x and its reflection through the median line. Since this median line must pass through the yolk, it must lie between the two lines T_1 and T_2 tangent to the yolk and perpendicular to L. Therefore, we know that x beats all points on L 'southwest' of x but is beaten by points on L 'northwest' of x and, more specifically, that the boundary of $W(x)$ intersects L somewhere between the points p_1 and p_2 that are the reflections of x through T_1 and T_2 respectively. Note that the distance between p_1 and p_2 is equal to $4r$.

It is evident that the distances from x to T_1 and T_2, and thus also to p_1 and p_2, are maximized when L is rotated clockwise so that it passes through the center of the yolk, in which event T_1 intersects L at a distance of $d - r$ from x and T_2 intersects L at a distance of $d + r$ from x. Therefore (1) the minimum distance that a point on this line can be from x and fail to beat x is $2(d - r)$, so its distance from the center of the yolk is $d - 2r$ and (2) the maximum distance that a point can be from x and still beat x is $2(d + r)$, so its distance from the center of the yolk is $d + 2r$. This gives us the following.

Proposition 9. Point x beats all points more than $2r$ further away from the center of the yolk than point x is, and x is beaten by all points more than $2r$ closer to the center of the yolk than x is.

On the other hand, if we rotate L counterclockwise so that it lies within the shaded cone labeled 'chaotic region' (bounded by lines perpendicular to the two lines passing through x that are tangent to the yolk), T_1 and T_2 intersect L on opposite sides of x, so the median line perpendicular to L may intersect it on either side of x and x may be beaten by points on L on either side of x (or, in the event the median line passes through x itself, x beats all points on L). Clearly the size of the 'chaotic region' (measured by the angle α, as shown in Figure 3) increases as the ratio of d to r decreases and if the ratio falls below 1 (putting x inside the yolk), it surrounds x, so that x may be beaten by points 'on all sides' and majority rule has no further centralizing tendency.[10]

[10] As the line L through x is rotated, the locus of points p_1 and p_2 trace out an inner and outer cardioid with center c, cusp x, and eccentricity $2r$, as established in Ferejohn et al. (1984) and McKelvey (1986). (The inner cardioid disappears when $d < r$.) If $r = 0$ (i.e., given Plott symmetry), the cardioids coincide with circle with center c and radius d, i.e., the preferred set with respect to x of the central voter as given by Corollary 8.2. Thus variation in the radius of the yolk makes Plott symmetry simply a limiting case, not a wholly distinctive case.

Given the definition of the yolk, there are three median lines tangent to the yolk that form the yolk triangle, as shown in Figure 2. Thus every point in the yolk — indeed, every point that lies within *any* triangle formed by three median lines [11] — not only may but must have a disorderly win set, with different majorities pulling towards each of the three median lines.

Any points x and the center of the yolk must lie on the same side of at least one of the three median lines forming the yolk triangle and, on any one of the three lines through x and perpendicular to one of these median lines, x is beaten by points up to its reflection x' through the median line. Since the median line is the perpendicular bisector of the line between x and x', any point (including c) on the same side of the bisector as x is closer to x than to x', giving us the following.

Proposition 10. In the absence of Plott symmetry, for any point x there is some other point that both beats x and is further from the center of the yolk than x is.

This fact drives McKelvey's theorem. By repeated application of Proposition 10, we can construct trajectory of this form: x is beaten by some point y further from the center of the yolk than x is; in turn, y is beaten by some point w further from the center of the yolk than y is; and so forth. Thus, we can construct such a trajectory from x to some point u at a distance greater than $d + 2r$ from the center of the yolk, where d is the distance from x to the center of the yolk. By Proposition 9, u is beaten by x. Generalizing, we have the following.

Proposition 11. In the absence of Plott symmetry, for any pair of points x and y there is a majority preference cycle including both x and y.

If two majority preference cycles have a point in common, there is a single cycle in the encompassing all the points in both cycles (see Black, 1958, p. 48), so we can make this further generalization.

Proposition 12 (McKelvey's [1979] Global Cycling Theorem). In the absence of an unbeaten point, a global cycle encompasses the entire alternative space.

While Propositions 9 and 10 support McKelvey's theorem, they also indicate that the trajectory leading from one point to another point considerably more distant from the center of the yolk will require many intermediate steps if the yolk itself is small, since each step in the trajectory can lead *at most* $2r$ further from the center of the yolk. Moreover, if this maximum outward movement of $2r$ per step is to be approached, the trajectory will jump wildly back and forth across the space, since — as Figure 3 indicates — the points most distant from the center of the yolk that beat a given point tend to be located on the far side of the yolk from that point.[12] Such considerations

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[11] This set of points has been dubbed the *heart* by Schofield (1999).

[12] Moreover, it is essential to McKelvey's result that the trajectory 'jump' over median lines; if the trajectory is required to proceed in a continuous fashion, its support by the majority associated with a given median line evaporates once the trajectory reaches the median line; this has the effect (in the two-dimensional case) of constraining the continuous trajectory essentially to the Pareto set. This is the import of a theorem due to Schofield (1978) that is often mistakenly conflated with McKelvey's theorem.

suggest significant limitations on the kind of 'agenda control' implied by McKelvey's theorem or on the claim that 'anything can happen' in the absence of Plott symmetry.[13]

4.4 The Uncovered Set

The allegedly 'chaotic' character of majority rule in two (or more) dimensions that suggests that social choice processes may 'wander all over the place' is anomalous because actual processes appear to be quite predictable and stable. In the face of this anomaly, formal political theorists have pursued two different, though not mutually exclusive, lines of inquiry.

The first, under the rubric of *structure-induced equilibrium* (Shepsle, 1979), recognizes that social choice is typically embedded in more or less elaborate institutional structure, which constrains choice so as to create (perhaps rather arbitrary) stable outcomes that would not exist under pure majority rule. A salient example is *dimension-by-dimension* voting, under which the outcome is the point that corresponds to the median induced ideal point in each dimension. Clearly such a point always exists and, by definition, it is not beaten by any points that propose change in one dimension only (as a 'germaneness rule' might require); however, in the absence Plott symmetry, it is beaten by points that propose changes in two (or more) dimensions simultaneously.[14]

The second, in contrast, looks for some deeper structure and coherence in majority preference that may constrain or guide political choice processes, even in the face of global cycling and independent of particular institutional arrangements. The *uncovered set*, first proposed by Miller (1980) in the finite alternative tournament setup discussed in Chapter 3.1 and then extended to the spatial context by McKelvey (1986), is one contribution to this line of theorizing.

Alternative *x covers y* if x beats y and x also beats every alternative that y beats.[15] Thus $W(x)$ is a proper subset of $W(y)$ and, being based on set inclusion, the covering relation is transitive and the *uncovered set*, comprised of all alternatives not covered by other alternatives, always exists. An uncovered alternative x has this strategically important *two-step property* — x beats any other alternative y in no more than two steps, i.e., either (i) x beats y or (ii) there is some third alternative z such that x beats z and z beats y (for otherwise y would cover x). Moreover, theoretical analysis shows that a variety of social choice processes (e.g., electoral competition, strategic voting under standard amendment procedure, cooperative voting with free coalition formation, open agenda formation, etc.) lead to outcomes in the uncovered set, as do various experimental and empirical results (see Chapter 6.2).

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13 This line of argument, anticipated by Tullock (1967), is pursued in Feld et al. (1989). The same considerations suggest that other 'chaotic' implications sometimes drawn from McKelvey's theorem may be overstated (e.g., Feld and Grofman, 1991 and 1996; Tovey, 2010).

14 Moreover, this outcome changes if the issue dimensions are rotated in the space (Feld and Grofman, 1988).

15 When ties are possible, complexities arise in the definition of covering that need not concern us here; see Penn (2006) and Duggan (2013). Note that Proposition 3(c) establishes that, given single-peaked preferences, covering and majority preference are equivalent.

Several points established in the finite alternative context (Miller, 1980) extend to the spatial context: the uncovered set is a subset of both the Pareto set and the top cycle set and it collapses to the Condorcet winner if one exists. However, in the spatial context, the second result is unhelpful for, as we have seen, either there is no top cycle (given Plott symmetry) or there is an all-encompassing top cycle (otherwise). In addition, McKelvey (1986) showed that the size and location of the uncovered set is related to the size and location of the yolk — namely, the uncovered set lies within a circle centered on c with a radius of $4r$. Little more was known about the uncovered set until computer programs were recently developed that can (approximately) demarcate its boundary given a configuration of ideal points (see Bianco, 2004, and Chapter 6.2; also see Miller, 2007).

If x covers y, the covering relationship manifests itself geometrically in the spatial context, with the boundary of $W(y)$ literally enclosing $W(x)$. As discussed in Miller (2007), this can occur in two ways. In its *proximate* mode, covering operates between neighboring points. In this event, both win sets have essentially the same shape, $W(x)$ being simply a slightly shrunken replica of $W(y)$. This requires a kind of unanimity among voters — specifically, a point y is covered by neighboring point x if and only if x is closer to the ideal points of every voter whose indifference curve through y demarcates part of the boundary of $W(y)$.

Point x most obviously proximately covers y if x Pareto-dominates y. Since x is unanimously preferred to y, x is closer to *every* ideal point than y is, and $W(x)$ is an everywhere shrunken replica of $W(y)$ and lies entirely inside it. Thus it is proximate covering that drive the uncovered set into the Pareto set. But proximate covering can operate within the Pareto set in special circumstances, namely if symmetries render some voters 'invisible' in that their indifference curves norm form the boundary of win sets. One such circumstance we have already considered: given Plott symmetry, Corollary 8.2 tells us that all win sets are identical to the preference sets of the central voter, so it is also proximate covering that shrinks the uncovered set to the central voter's ideal point, i.e., the Condorcet winner.[16]

In its *distant* mode, x covers y only if the two points are some distance apart. In this case, $W(x)$ is not simply a shrunken replica of $W(y)$. Rather the win sets may be quite differently shaped but $W(x)$ is sufficiently smaller than $W(y)$ that its different shape can be enclosed within the boundary of $W(y)$. A natural question to ask about such covering is how great a distance is required. What really matters is that x must be substantially closer to the center of the yolk than y is. Applying Proposition 9 and the two-step property tell us that x covers at a distance any point y more than $4r$ further from the center of the yolk than x is. However, x may cover y at a distance even if y is considerably less that $4r$ further from c than x is.

The *uncovered set of point* x, $UC(x)$, is the set of points not covered by x. Since $W(x)$ is the set of all points that beat x in one step and $UC(x)$ is the set of all points that beat x in one or two steps, $UC(x)$ is a superset of $W(x)$. The set of points $UC(c)$ not covered by the center of the yolk is of particular significance. W(c) reaches to three points at a distance of $2r$ from c, i.e., the reflections

[16] Even in the absence of Plott symmetry, Miller (2007) show that some ideal points may line up in a way that renders some voters invisible with respect to the demarcation of win sets, permitting some proximate covering within the Pareto set.

of c through each of the median lines forming the yolk triangle), but to no more distant points (since every other median line passes closer to c). By Proposition 9, the win sets of these three points extend no more than $2r$ further from c, so $UC(c)$ extends outward to a distance of no more $4r$ from c.

The uncovered set is the set of all points not covered by *any* other point — that is, the intersection of the sets $UC(x)$ for *all* points x in the space. Therefore, the uncovered set is a subset of $UC(c)$ and lies within the same bound, giving us the $4r$ bound on the uncovered set first identified by McKelvey (1986). But this raises two questions: is the $4r$ bound on $UC(c)$ overgenerous and is the uncovered set itself only a bit smaller than $UC(c)$ or is it considerably smaller? (A third and more fundamental question also arises, which we take up in the following section: how large typically is the yolk itself relative to the configuration of ideal points?)

On the basis of an examination of many ideal point configurations using computer software, Miller (2007) reached the following conclusions. For points x close to c, the $UC(x)$ sets tend to be irregularly shaped with 'peninsulas' emanating from a central core. Their central cores substantially coincide, but their 'peninsulas' emanate in offsetting directions. As the uncovered set is formed from the intersection of such sets, the 'peninsulas' are snipped off, leaving an uncovered set that is essentially central core of all sets $UC(x)$, which in turn is essentially the central core of $UC(c)$. As a result, the uncovered set is considerably more compact than the individual $UC(x)$ sets, with a boundary generally lying about $2r$ to $2.5r$ from c (unless proximate covering further reduces its size). If the uncovered set is important theoretically or (as is argued in Chapter 6.2) empirically, we have another reason why the location and size of the yolk are important.

4.5 The Size and Location of the Yolk

A final question pertains to the typical size of the yolk, relative to the Pareto set or some other measure of the span of the ideal point configuration. It is also important to know whether the yolk typically occupies a 'central' location (e.g., relative to the Pareto boundary) or whether it may be substantially 'off-center.'

When the concept was first propounded, there was a widespread intuition that the yolk tends to shrink in size as the number and diversity of ideal points increase. However, it was difficult to confirm this intuition or even to state it in a theoretically precise fashion. Feld et al. (1988) took a few very modest first steps. Tovey (2010) took a considerably larger step by showing that, if ideal point configurations are random samples drawn from a 'centered' continuous distribution, the expected yolk radius approaches zero as the number of ideal points increases without limit.[17] But Tovey's theoretical result left two important questions open: the rate at which the yolk shrinks as the number of ideal points increases, and the impact of clustering such as we might expect to see in empirical ideal point data (and certainly do see in figures in Chapters 6.1 and 6.2), on the size and location of the yolk.

[17] In fact, Tovey's paper had been written and circulated 20 years prior to publication.

Early simulations by Koehler (1990), subsequently extended by Hug (1999), together with more recent simulations by Brauninger (2007), show that the expected size of the yolk declines quite rapidly as larger samples of ideal points are drawn out of a bivariate uniform distribution. More recently, Miller (2007) replicated these findings for bivariate uniform and (more natural appearing) normal distributions. Once a low threshold of fewer than a dozen ideal points is crossed, the expected yolk radius shrinks as the number of voters increases, and given configurations of several hundred voters, the expected yolk radius is about one-quarter (and yolk area about 6%) of that for most small configurations. For larger configurations, the expected yolk radius appears to follow an inverse square root law with respect to sample size (in the manner of sampling error more generally), so that the expected yolk area follows a simple inverse law. Proposition 9 then implies that, given large 'random' ideal point configurations, win sets of points at some distance from the very small yolk come very close to forming perfect circles (as if Plott symmetry existed). Figure 4 provides example for $n = 435$.[18] A figure like this leads one to wonder whether excessive ink has been spilled decrying the chaotic nature of majority rule in typical two-dimensional voting games. It is clear that in such a configuration, the known bounds on the uncovered set confines it to a tiny area within the Pareto set.

However, with respect to the second question, it is also evident that clustering of ideal points can greatly increase the expected size of the yolk and also skew it off-center. Figure 5 shows 435 ideal points distributed in clusters of virtually equal size (218 on the left and 217 on the right). (This closely resembles a typical distribution of D-NOMINATE scores in the contemporary U.S. House of Representatives.) Note that almost all median lines form a ''bow tie" pattern and pass through a small area about midway between the two clusters. This might suggest that the yolk lies in this small central region, but there must be at least one additional median line (in Figure 5 there are two) that lies more or less vertically across the centrist face of the majority cluster, with the entire minority cluster and the empty space between them on the other side. Since it intersects all median lines, the yolk must lie within the majority side of the bow tie (which essentially forms the yolk triangle); as such, it is much larger than in Figure 4 and is non-centrally located, being nestled against the centrist face of the majority cluster.[19] In such circumstances, therefore, the yolk is not centrally located within the configuration of ideal points. Moreover, it is evident that if just one ideal point were moved from the left cluster to the right, the yolk would flip from one side of the bow tie to the other.

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[18] Figure 4 and 5 were generated by *CyberSenate*, a computer program for analyzing spatial voting games developed by Joseph Godfrey.

[19] It is not centrally located with respect to either the Pareto set or the mean ideal point position; of course, the yolk generally track the 'center' in the sense of the median. Note that, if the vertical spread of ideal points were compressed relative to the horizontal polarization, the yolk would become smaller and even more conspicuously pushed in the majority direction.

5. Multiple Dimensions and k-Majority Rule

In the brief concluding section, we give some consideration to voting rules more demanding than simple majority with an odd number of voters, which leads to some consideration of dimensions beyond two.

In the two dimensional case, once we move beyond simple majority rule with n odd to a more demanding voting rule, median lines become 'thick'; put otherwise, they turn into 'pivotal strips' whose width is the distance between pivotal induced ideal points on lines perpendicular to the strip. Thus $n - k$ ideal point lie on either side of the strip and $2k - n$ ideal points lie on the strip. Any point x in the strip is unbeaten by points on the line through x and perpendicular to the strip. But (in the manner of Proposition 7) any point lying off a strip is beaten under k-majority rule by the closest point on the strip, so unbeaten points exist if and only if all such strips have a common intersection. Other things being equal, pivot strips (with positive width) are more likely to have a common intersection than median lines (of zero width), the more so the thicker the strips are. Moreover, the more demanding the voting rule is, the thick pivot strips are likely to be.

Let's first consider the case of majority rule with n even. In the one-dimensional case, this typically produces a small interval of unbeaten points but, in the event that ideal points $x^{n/2}$ and $x^{n/2+1}$ coincide, their common ideal point is the only unbeaten point (and a Condorcet winner). In the two dimensional case, some lines pass through two ideal points and with $(n/2) - 1$ ideal points on either side, and as such are pivotal strips of zero width. Since these lines must have a common intersection for an unbeaten point to exist, essentially the same pairwise symmetry condition must hold as in the case of n odd and, with six of more voters (requiring three or more lines must have a common intersection), this is unlikely. But in the event they do have a common intersection, it typically will not coincide with a voter ideal point and, while (uniquely) unbeaten, it is not a Condorcet winner. In the special case of four voters, an unbeaten point must exist: if their ideal point all lie on the boundary of the Pareto set, there are just two pivotal strips of zero width and they must intersect; if one ideal point x^i lies in the interior of the Pareto triangle, there are three pivotal strips of zero width all passing through x^i and thus having a common intersection.

We now consider how demanding k-majority rule can be and still fail to guarantee the existence of a core in two-dimensions. This requires (at least) three pivotal strips that fail to have a common intersection. Consider the triangle formed by their 'inside' boundaries B_1, B_2, and B_3; thus at least k ideal points lie 'outside' B_1, B_2, and B_3. Some simple calculations show that this cannot hold if $k > 2n/3$; but it can hold if $k = 2n/3$ provided exactly one-third of the ideal points are located at or 'outside' each vertex of the triangle and thus on or 'outside' two of the three boundary lines. Thus we have the following proposition.

Proposition 13. Given a two-dimensional alternative space, there is a non-empty core under any k-majority rule with $k > 2n/3$, but given $k \leq 2n/3$, the core may be empty for some configurations of ideal points.[20]

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[20] Note that, with three voters, simple majority rule is also 2/3 rule and that, with four voters, simple majority rule is also 3/4 rule.

Clearly if k is close to $2n/3$, a core exists except given highly contrived and atypical configurations approximating that discussed just above, and as k approaches simple majority rule the non-existence of unbeaten points becomes more and more likely. Given the distribution shown in Figures 4, a core exists with k as small as 230; more surprisingly, the same is true with the distribution shown in Figure 5.

Proposition 13 can be generalized to higher dimensions. Given three dimensions, the three lines B_1, B_2, and B_3 enclosing a (plane) triangle with three vertices become four planes enclosing a (solid) tetrahedron with four vertices, so a core is not assured until k = $3n/4$. Generalizing further, we get the following.

Proposition 14 (Greenberg's [1979] Core Existence Theorem). Given an m-dimensional space, there is a non-empty core under any k-majority rule with $k > (m/m{+}1)n$, but given $k \leq (m/m{+}1)n$, the core may be empty for some configurations of ideal points.

In general, more demanding voting rules make a non-empty core more likely, while more dimensions make a non-empty core less likely.

Finally, we note that making voting institution more complex, e.g., having a bicameral rather than unicameral voting body has an effect similar to making a voting rule more demanding. For example, in the one-dimensional case, a unicameral body using simple majority rule has a Condorcet winner, i.e, the ideal point of its median voter; a bicameral body in which each chamber uses simple majority rule has a more extensive core, i.e., the interval between the median ideal points of the two chambers. Likewise, in the two-dimensional case, a unicameral body using simple majority rule almost always fails to have a core, but a bicameral body in which each chamber uses simple majority rule has a non-empty core in many cases. This kind of analysis of the spatial model is pursued in Chapter 4.3.

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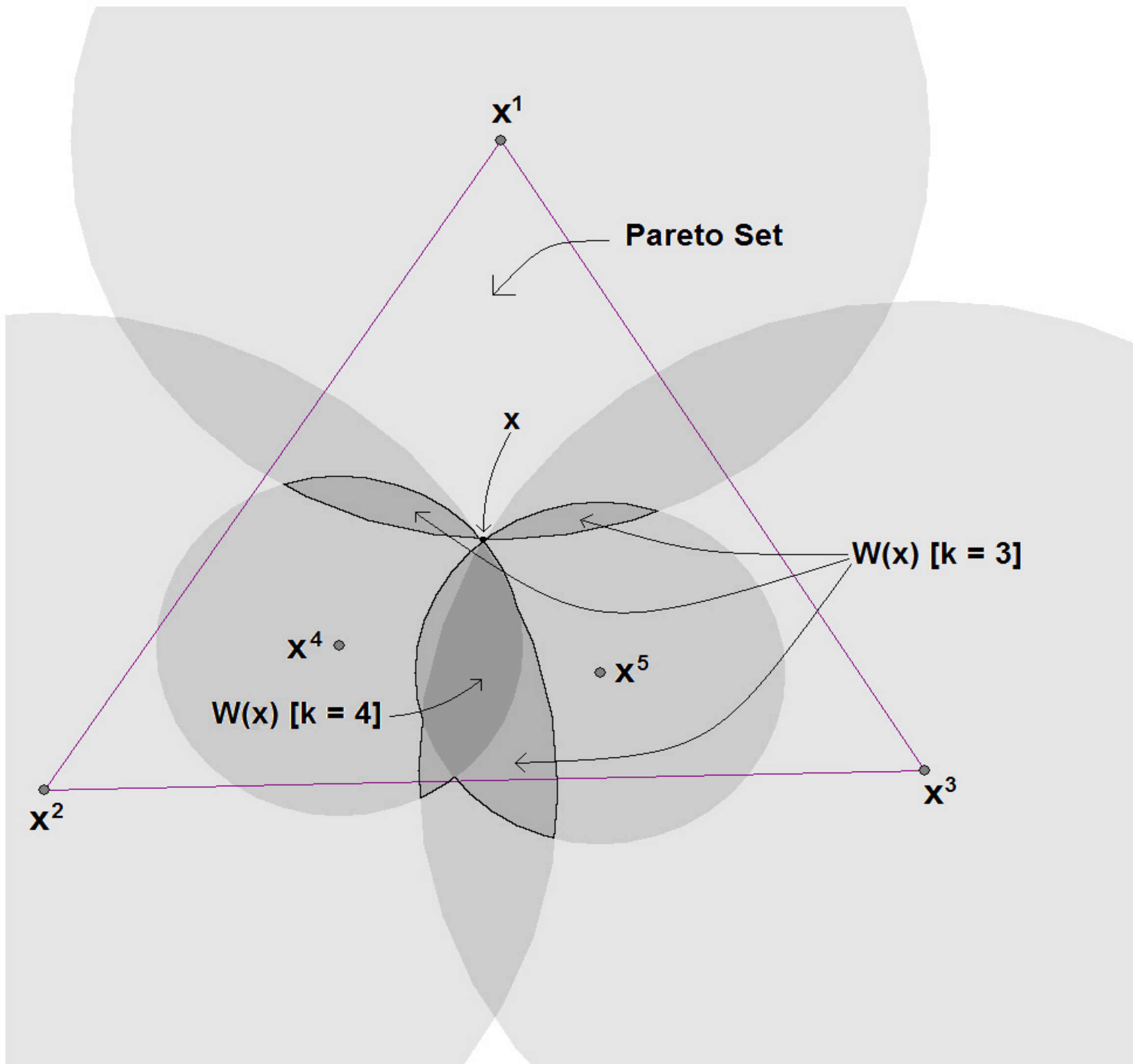

Figure 1. A Win Set in Two Dimensions

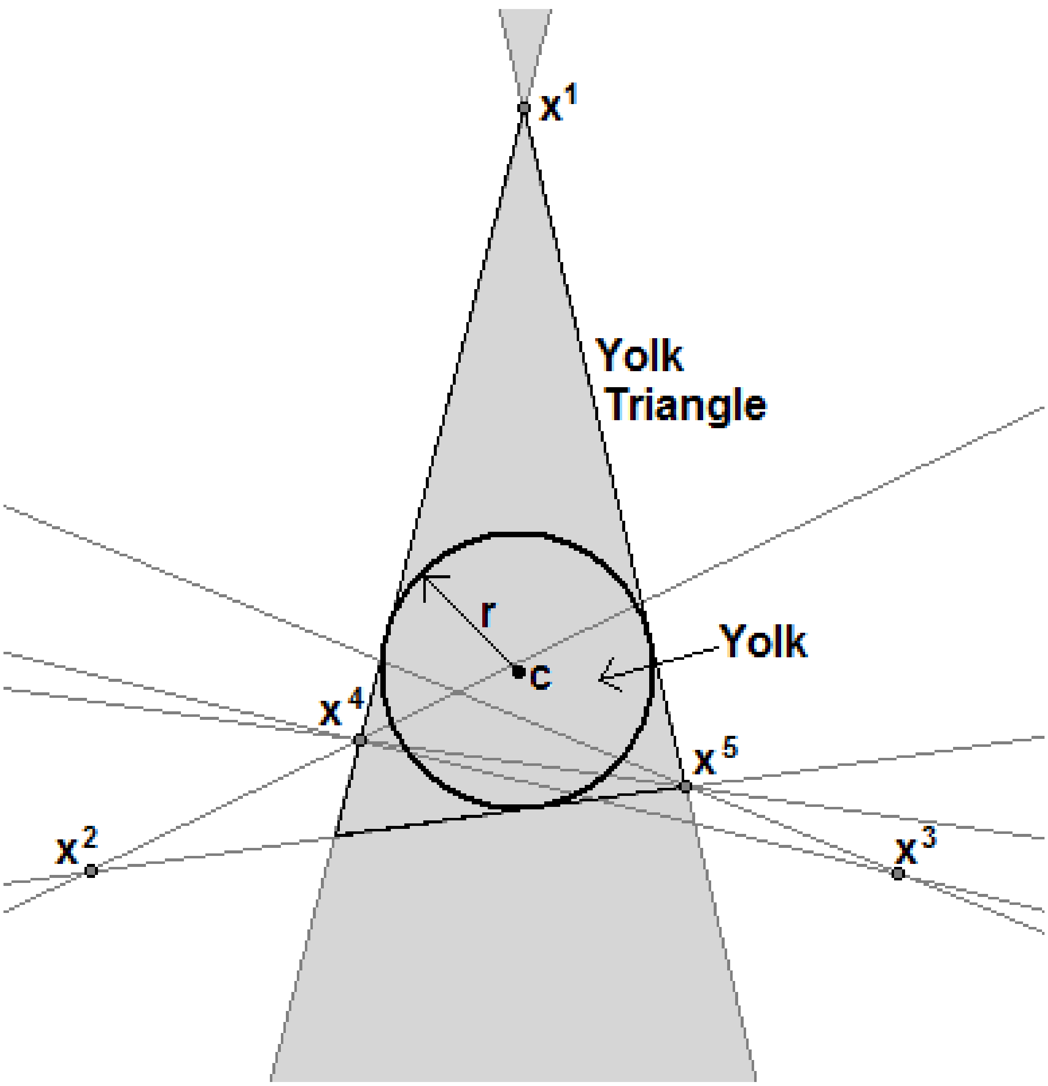

Figure 2. Median Lines and the Yolk

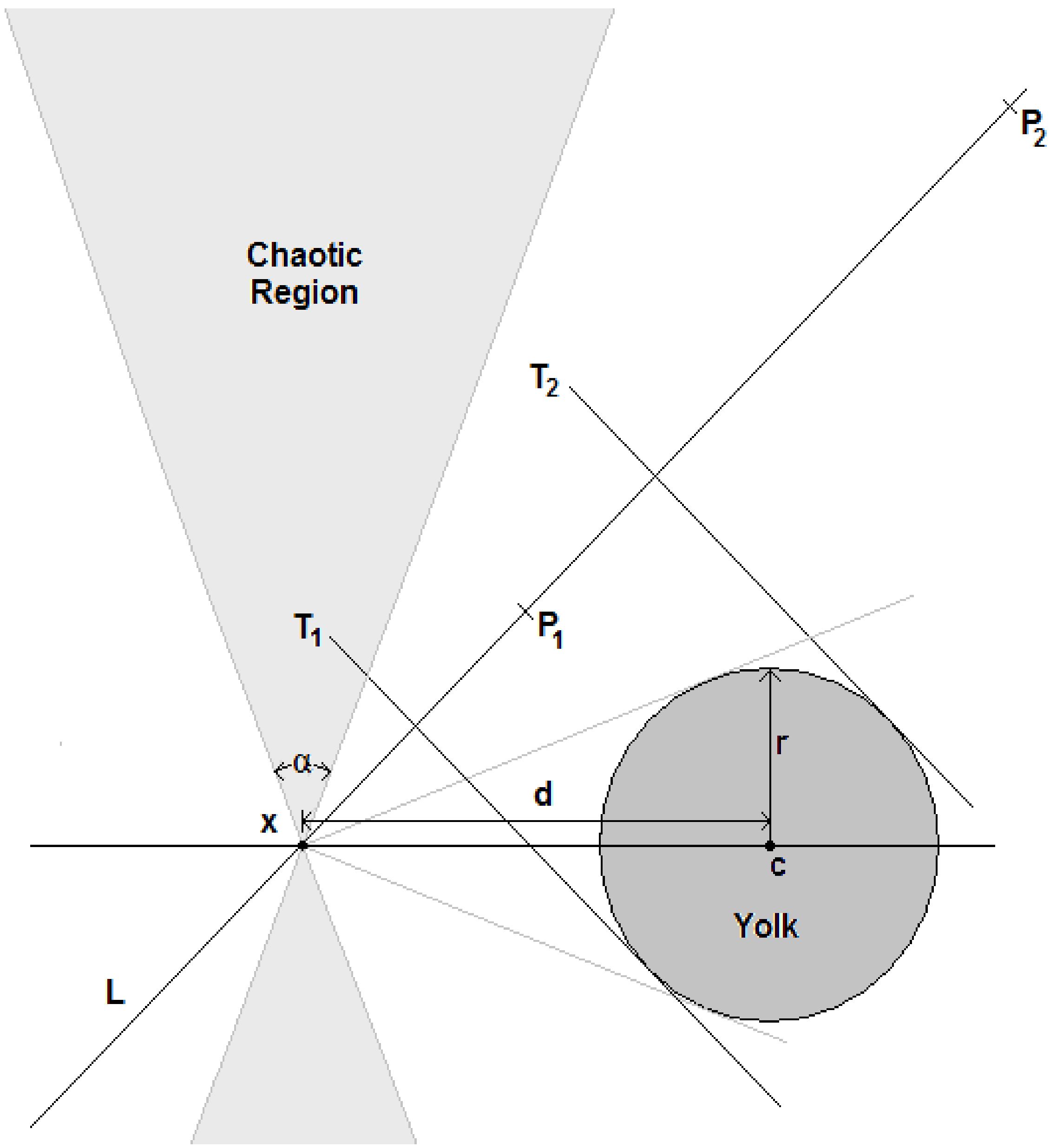

Figure 3. The Yolk and Bounds on Win Sets

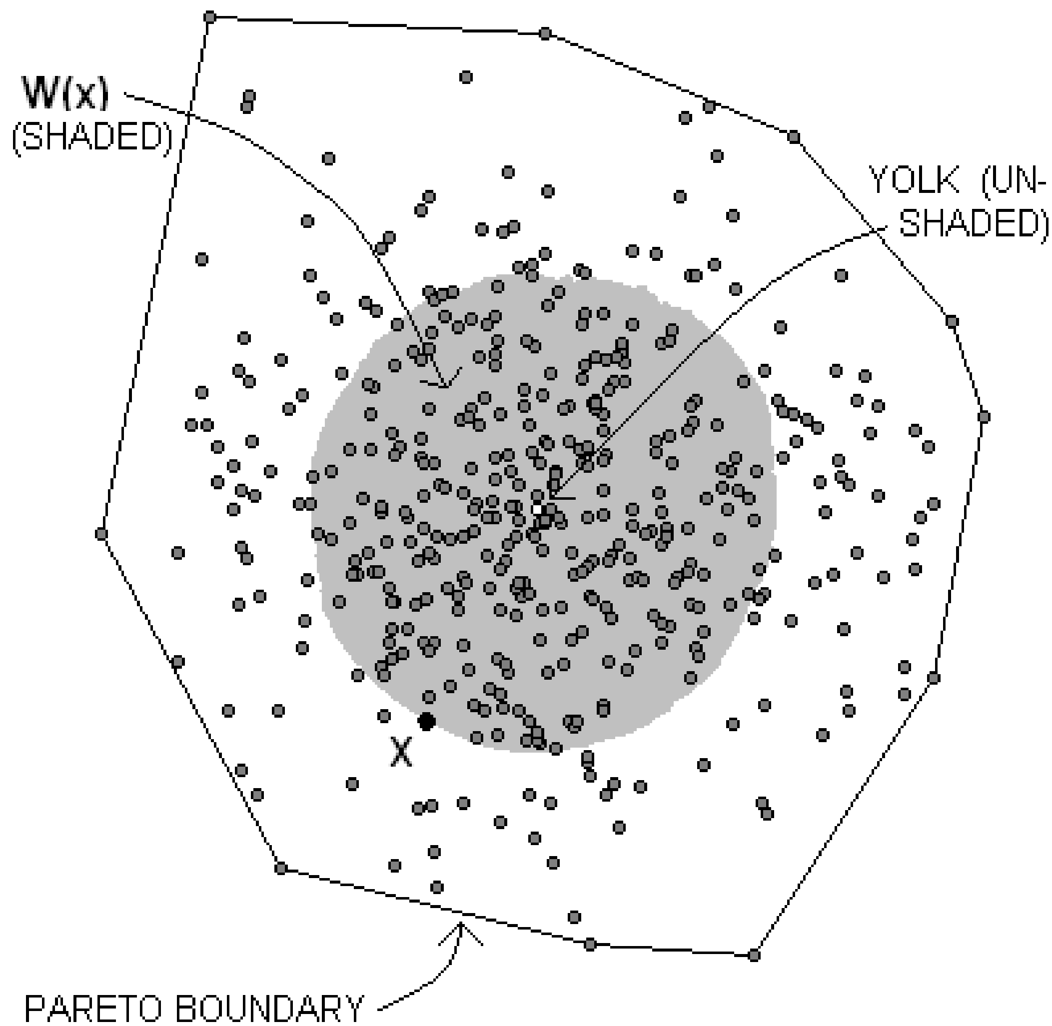

Figure 4. The Yolk and a Win Set with n = 435 in an Unclustered Distribution

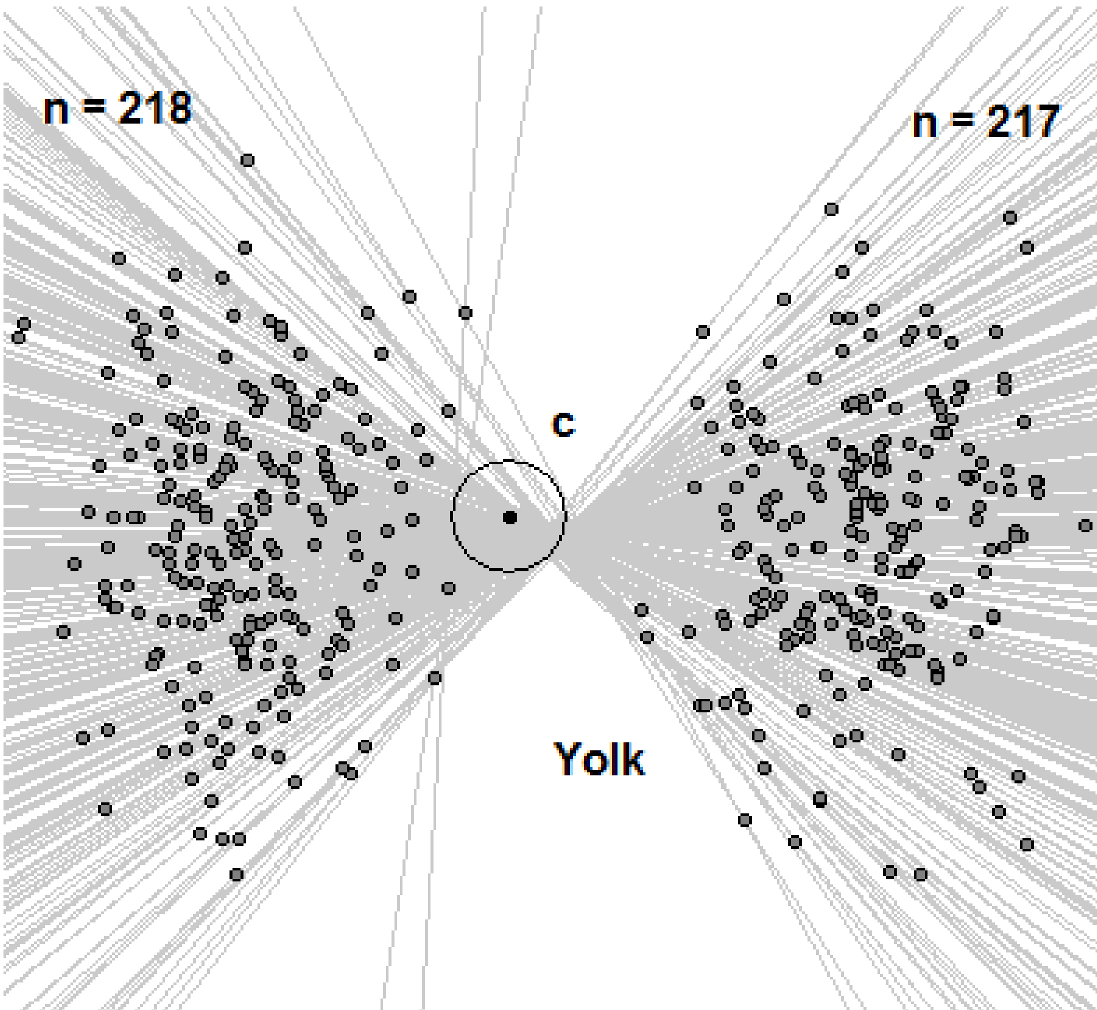

Figure 5. Median Lines and the Yolk with n = 435 in Polarized Clusters