PLURALISM, SEPARABLE PREFERENCES,

AND SOCIAL CHOICE

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PLURALISM, SEPARABLE PREFERENCES, AND SOCIAL CHOICE

This paper reviews, clarifies, and elaborates some points made in Feld and Grofman (1991), Lin (1991), and Miller (1983) concerning links between cyclical majorities, permanent (or universal) winners, losers, and majorities, and coalitions of minorities. Standard definitions (e.g., preference profile, majority preference, single-peaked preferences, median voter, Condorcet alternative, separable preferences), notation (e.g., x P y), and results (e.g., the Median Voter Theorem) are assumed. However, basic terminology and results concerning separable preferences are presented, since these are less standard.

1. The Unrestricted Model

Feld and Grofman work within what we may call the unrestricted model, which is set up as follows.

The Unrestricted Model:

(a) a finite set X of three or more alternatives;

(b) a profile of unrestricted strong voter preference orderings over the alternatives in X; and

(c) the number of voters is odd; so

(d) majority preference is strong (a tournament).

For a given preference profile, a winner is a voter who is a member of some effective coalition C, i.e., some majority coalition C such that x P_i y for all i in C (with the result that x P y); in other words, a winner supports the winning side on some pair of alternatives. A loser is a voter who is a member of the complement of some effective coalition; in other words, a loser opposes the winning side on some pair of alternatives. At most preference profiles, of course, most (or all) voters are both winners (with respect to some pairs of alternatives) and losers (with respect to other pairs of alternatives).

For a given preference profile, a permanent winner (in X) is a member of every effective coalition and so is a winner on every pair of alternatives in X. A permanent loser (in X) is a member of no effective coalition and so is a loser on every pair of alternatives in X. A permanent majority (in X) is a set of voters of majority size all of whom are permanent winners (in X).

Note that the existence of permanent winners, losers, and majorities depends only on the preferences profile over X and not on any differences in the "power" of voters or of equal-sized coalitions, since by focusing on majority rule we are assuming simple majoritarian power relations in any event.

OBSERVATION 1. All permanent winners, and thus all members of a permanent majority, have identical preferences over all alternatives in X.

OBSERVATION 2. All permanent losers have identical preferences over all alternatives in X.

OBSERVATION 3. Permanent winners and permanent losers have directly opposed preferences, i.e., if i is a permanent winner and j is a permanent loser, then $x P_i y$ if and only if $y P_j x$ for all x and y in X.

PROPOSITION 1. If (a) there is a permanent winner in X, or (b) there is permanent loser in X, majority preference over X is transitive.

Proof. If there is a permanent winner i, majority preference is identical to P_i; if there is a permanent loser j, majority preference is the reverse of P_j. In either event, the transitivity of majority preference follows from the transitivity of individual preference.

Thus the existence of a permanent winner or loser precludes cycles. Put the other way around, the existence of a (top or non-top) cycle implies the absence of both permanent winners and losers.

We may ask whether the converse of Proposition 1, or either part of it, is true. The answer in general is no.

PROPOSITION 2. If X has four or more alternatives, majority preference may be transitive even in the absence of both permanent winners and permanent losers.

Proof. Both points may be demonstrated by a simple example.

EXAMPLE 1.

P_1	P_2	P_3	P
x	y	z	v
v	v	v	y
y	x	y	x
z	z	x	z

In Example 1, there are three voters with preference orderings P_1, P_2, and P_3; majority preference P is transitive but, at the same time, there is no permanent winner or loser. (Everyone loses on the pairing of v with his most preferred alternative, but everyone wins on the pairing of v with each remaining alternative.)

In the special case three alternatives, the converse of (a), but not of (b), in Proposition 1 holds. (Of course, the question of transitive versus cyclical preference does not arise if there are just two alternatives.)

PROPOSITION 3. Given X has just three alternatives, if majority preference over X is transitive, there is a permanent winner in X.

Proof. Let X = {x,y,z} and suppose without loss of generality that x P y, y P z, and x P z. Let the effective coalitions for these three majority preference relationships be C, C', and C" respectively. C and C', being majority coalitions, must intersect. Let i belong to this intersection;

thus x P_i y and y P_i z. By the transitivity of individual preference, x P_i z, so i also belongs to C" and is a permanent winner.

COROLLARY 3.1. Given X has just three alternatives, if there is a permanent loser in X, there is also a permanent winner.

Proof. However many alternatives there are in X, by Proposition 1, if there is a permanent loser, majority preference is transitive. Given three alternatives in X, by Proposition 2, if majority preference is transitive, there is a permanent winner.

It is worth noting that, apart from the relationship given in Corollary 3.1, permanent winners and permanent losers are completely independent.

PROPOSITION 4. Given an arbitrary preference profile, we may find

(a) both a permanent winner and a permanent loser; or

(b) a permanent winner but no permanent loser; or

(c) a permanent loser but no permanent winner (if there are four or more alternatives); or

(d) neither a permanent winner nor a permanent loser.

Proof. It is sufficient to provide an example for each case (a) - (d).

(a) This would occur if the group were polarized into just two subsets (necessarily of unequal size, since n is odd), with identical preference orderings within each subset and directly opposed preference ordering between the subsets. Each member of the larger subset is a permanent winner, and each member of the smaller subset is a permanent loser.

(b) This would occur if there were complete unanimity in the group; every voter is a permanent winner.

(c) This would occur if the preference profile in Example 1 were modified in this way: two voters have each if the listed individual preference orderings and a seventh individual has an ordering the reverse of the listed majority ordering. It remains true that there is no permanent winner; at the same time the majority ordering is unchanged, so the seventh individual is a permanent loser.

(d) By Proposition 1, this would occur whenever majority preference is cyclical; but it may also occur when majority preference is transitive (recall Example 1 and see Example 2 below).

2. The Single-Peaked Model

We now consider the implication of single-peakedness for permanent winners, losers, and majorities.

The Single-Peaked Model:

(a) a finite set X of three or more alternatives;

(b) a profile of strong single-peaked voter preferences over the alternatives in X; and

(c) the number of voters is odd; so

(d) majority preference is a strong ordering.

Single peakedness implies the existence of a strong ordering of the alternatives (e.g., from "left" to "right"). Let $x < y$ mean x comes before (e.g., "is to the left of") y in this ordering.

We introduce several useful but non-standard definitions. Given single-peakedness, an alternative x is interior in X if there are two other alternatives y and z in X such that $y < x < z$. An alternative is extreme if it is not interior. Given that all alternatives in X are distinct, precisely two of them (i.e., the first and last in the strong ordering) are extreme. An voter is extreme if his most preferred alternative is extreme.

OBSERVATION 4. The least preferred alternative of each voter is extreme.

OBSERVATION 5. Extreme voters with the same first preference have identical preference orderings.

OBSERVATION 6. Extreme voters with different most preferred alternatives have directly opposed preference orderings.

Observations 5 and 6 imply that the set extreme voters may be partitioned into two subsets with opposite preference orderings. However, one or both subsets may be empty.

While single peakedness precludes cycles, it does not assure the existence of permanent winners or losers.[1] We can spell out the connection between single peakedness and permanent winners and losers more precisely. First, we give the following preliminary result.

PROPOSITION 5. A median voter

(a) wins on every pair of alternatives one of which is his first preference in X; and

(b) wins on every pair of alternatives both of which lie on the same side of his most preferred alternative; but

(c) may lose on pairs of alternatives that lie on opposite sides of his most preferred alternative.

<u>Proof</u>.

(a) By the Median Voter Theorem, a median voter i's most preferred alternative is the Condorcet alternative x. So $x \, P_i \, y$ and $x \, P \, y$ for all y in X other than x.

(b) Consider the Condorcet winner x and any two other alternatives y and z that lie on the same side of x with y between x and z. By single peakedness, $y \, P_i \, z$ for a median voter or any voter whose most preferred alternatives lies on the opposite side of x from y and z. By the definition of a median voter, a majority of voters have such a preference, so also $y \, P \, z$.

(c) This is demonstrated by the following example.

EXAMPLE 2.

$\underline{P_1}$	$\underline{P_2}$	$\underline{P_3}$	$\underline{P}$
x	y	z	y
y	z	y	z
z	v	x	x
v	x	v	v

The strong ordering of alternatives implied by single peakedness is x < y < z < v (or the reverse). Voter 2 is the median voter (and y is the Condorcet winner) but 2 loses on the pair x and v.[2]

The following proposition summarizes the implications of single peakedness for permanent winners, losers, and majorities.

PROPOSITION 6. If preferences are single-peaked,

(a) only a median voter may be a permanent winner; but

(b) there may be no permanent winner (if X has four or more alternatives);

(c) a median voter is a permanent winner if he is extreme;

(d) only an extreme voter can be a permanent loser;

(e) there is a permanent loser only if median voters are extreme; and

(f) if median voters are extreme, they constitute a permanent majority.

Proof.

(a) Consider any voter i who is not median and whose most preferred alternative in X is y, necessarily distinct from the Condorcet winner x (for otherwise i would be a median voter). Since x P y, i cannot be a permanent winner.

(b) This is demonstrated by Example 2. (By Proposition 3, of course, there must be a permanent winner if there are just three alternatives.)

(c) By Proposition 5, a median voter may lose only on pairs of alternatives that lie on opposite sides of his most preferred alternative. But if a median voter is extreme, all other alternatives must lie on the same side his most preferred alternative.

(d) Consider any non-extreme voter i with a most preferred alternative x. Thus there are alternatives y and z such that y < x < z and x P_i y and x P_i z. If i is a median voter, x is the Condorcet alternative and x P y and x P z. Otherwise, let v be the Condorcet alternative. If x < v, then x P y; and if v < x, then x P z. In any event, non-extreme voter i wins on at least one pair and cannot be a permanent loser.

(e) Suppose there is a permanent loser i. Voter i's preference ordering is therefore the reverse of the majority ordering. The least preferred alternative in i's ordering, which by Observation 4 is extreme, is therefore the Condorcet alternative and thus the first preference of each median voter. Thus each median voter is extreme.

(f) Now suppose a median voter's most preferred alternative x is extreme. Since x is both extreme and the Condorcet alternative, an absolute majority of voters must be both extreme and median. By Observation 6, they have identical preferences. Thus they constitute a permanent majority.

We now give a non-standard definition. A single-peaked preference profile is <u>symmetric</u> if, for any pair of voters i and j with most preferred alternatives x_i and x_j respectively and for any pair of alternatives x' and x", (i) if x' < x_i < x_j < x", then x' P_j x" only if x' P_i x" and likewise (ii) if x' < x_j < x_i < x", then x' P_i x" only if x' P_j x". The intuition is this: if (i) if x' is "closer to" x_j than x" is (so that x' P_j x"), x' is certainly "closer to" x_i than x" is (so that x' P_i x"), and likewise for (ii). Geometrically (and this is the standard definition), if the alternatives in X are points along a continuum, the voter utility functions over the continuum that induce the preference orderings over X are not only single peaked but also symmetric about each ideal point.[3] Note that the preference ordering in Example 2 are not symmetric single-peaked; in order to induce the preference orderings shown, 2's utility must fall off more steeply to the left (where x is) of his ideal point than to the right (where v is), while the reverse is true for voter 3.

For the case of symmetric single-peaked preferences, Proposition 6 may be strengthened as follows.

PROPOSITION 7. If preferences are symmetric single-peaked, there is a permanent winner.

Proof. Of course, by Proposition 6, such permanent winner is a median voter. But, if there are several median voters, they may not all be permanent winners. It is necessary to show that (c) of Proposition 5 does not hold for at least one median voter in the special case of symmetric single-peaked preferences. Consider a median voter i with first preference x and any two alternatives y and z such that $y < x < z$. Suppose without loss of generality that $y \, P_i \, z$. We need to show that either (i) $y \, P \, z$ or (ii) there is another median voter i' such that $z \, P_{i'} \, y$ and $z \, P \, y$. For any voter k with most preferred alternative v such that $v \leq y$, $y \, P_k \, z$ follows from (general) single peakedness. For any voter j with most preferred alternative w such that $y < w < x$, $y \, P_k \, z$ follows from the definition of symmetric single peakedness. By the definition of a median voter, the set of all voters with any first preference u such that $u < x$, together with all median voters, constitutes a majority of voters. Thus either $y \, P \, z$ or there is at least one median voter i' (still with first preference x, of course) for whom $z \, P_{i'} \, y$ and by parallel argument $z \, P \, y$. In either event, there is a permanent winner.[4]

That some median voters are permanent winners and others are not (both within the same preference profile or in different preference profiles) really suggests that the notion of a "permanent winner" (as here defined) is not very significant politically. For the fact is that we expect all median voters in all preference profiles to be "totally satisfied," in that their most preferred alternative, i.e., the Condorcet winner, can be expected to be the social choice. The fact the some median voters may lose a few skirmishes along the way -- i.e., pairwise choices between alternatives neither of which will ultimately prevail -- and thereby fail to be "permanent winners," does not seem too significant. It was for this reason that Miller's argument was set up in a different framework (implicitly in the text, explicitly in one footnote; p. 737). We examine this model in the following Section 4. Before that we discuss "separable preferences."

3. Separable Individual Preferences

Suppose voters face a number of distinct "issues." Each issue comprises a set of positions. To simplify matters and to connect more directly with notions of majorities and minorities, we suppose that each issue is dichotomous, i.e., has just two positions. The set of alternatives is the set of all possible combinations of positions, one for each issue.[5] Given k dichotomous issues, it follows that there are 2^k alternatives.

As before, let X designate the set of alternatives and let $X_1, X_2,..., X_k$ designate the k different issues. Let the positions on a representative (dichotomous) issue X_h be x_h and $\overline{x}_h$. Thus the set of all alternatives X is the Cartesian product $X = X_1 \times X_2 \times ... \times X_k$. A typical alternative is $x = (x_1, \overline{x}_2,..., \overline{x}_h,...,x_k)$, and we say that x gives positions x_1, $\overline{x}_2$, etc.

Let $K = \{1,2,...,k\}$ be the set of issues, and consider any subset $A \subset K$ of issues. Let x_A be a partial alternative with respect to A, i.e., a specification of positions for all issues in A; and let X_A be the set of all such partial alternatives, i.e., the Cartesian product $X_A = \times X_h$ for all h in A. Then a complement $x_{\overline{A}}$ of x_A is a partial alternative with respect to the complementary set $\overline{A} = K - A$ of issues, i.e., a specification of positions for all issues in K but not in A; and let $X_{\overline{A}}$ be the set of all such complementary partial alternatives, i.e., the Cartesian product $X_{\overline{A}} = \times X_h$ for all h in K but not A. Thus any $(x_A,x_{\overline{A}})$ is an alternative. Let $(X_A,x_{\overline{A}})$ designate the set of all alternatives specified by the set of partial alternatives X_A and the complement $x_{\overline{A}}$ -- that is, the set of alternatives that differ only with respect to positions on issues in A, having the fixed complement $x_{\overline{A}}$.

We suppose that each voter has a separable preference ordering over the 2^k alternative, which may be defined as follows. Consider voter i's preferences over all alternatives in any set $(X_A,x_{\overline{A}})$. Now consider voter i's preferences over all alternatives in the set $(X_A,x'_{\overline{A}})$ for any different complement $x'_{\overline{A}}$. Voter i's preferences are separable by issues if and only if these preferences are identical, in that $(x_A,x_{\overline{A}})\ P_i\ (x'_A,x_{\overline{A}})$ if and only if $(x_A,x'_{\overline{A}})\ P_i\ (x'_A,x'_{\overline{A}})$ for any x_A and x'_A in X_A and for any $x_{\overline{A}}$ and $x'_{\overline{A}}$ in $X_{\overline{A}}$.

A voter with a preference ordering over alternatives that is separable by issues thus has "induced" preferences over partial alternatives (that remain the same regardless of how the remaining issues are resolved). And since individual issues positions correspond to partial alternatives with respect to some one-element set A, a voter with a preference ordering over alternatives that is separable by issues has induced preferences over positions on any individual issue (that remain the same regardless of how all other issues are resolved), so that for any issue X_h we can say that either $x_h \, P_i \, \overline{x}_h$ or $\overline{x}_h \, P_i \, x_h$, i.e., that either x_h or $\overline{x}_h$ is i's more preferred position. (We continue to assume that all preferences are strong.)

We now turn the matter around and ask, if we know voter i's separable preferences on each issue, what can we infer about his preference ordering over all alternatives? The answer is that we can infer a good deal but not everything.

Consider the simplest example with just two issues, X_1 and X_2. For an arbitrary voter i, let us label the issue positions so that $x_1 \, P_i \, \overline{x}_1$ and $x_2 \, P_i \, \overline{x}_2$. There are four alternatives and thus six pairs of alternatives between which i has preferences.

EXAMPLE 3. (x_1,x_2) $(x_1,\overline{x}_2)$ $\underline{P_i}$ $\underline{P'_i}$

(x_1,x_2) (x_1,x_2)
$(x_1,\overline{x}_2)$ $(\overline{x}_1,x_2)$
$(\overline{x}_1,x_2)$ $(x_1,\overline{x}_2)$
$(\overline{x}_1,\overline{x}_2)$ $(\overline{x}_1,\overline{x}_2)$

$(\overline{x}_1,\overline{x}_2)$ $(\overline{x}_1,x_2)$

Preference relationships (1)-(4) follow directly from the definition of separability. Given preference relationships (1) and (3) -- or (2) and (4), preference relationship (5) follows from the transitivity of individual preference. However, we can infer nothing about preference relationship (6) from i's issue preferences. Voter i's preference between $(x_1,\overline{x}_2)$ and $(\overline{x}_1,x_2)$ depends on how i would "trade off" a preferred position on one issue for a preferred position on another issue -- put otherwise, how "intense" his preference for x_1 is compared with his preference for x_2 -- something we cannot infer knowing only i's position on the individual issues. So we know i's overall preference ordering is either P_i or P'_i but we don't know which.

Let us generalize and formalize these considerations. Given that voter i's preferences are separable by issues, we can partition all pairs of alternatives initially into two subsets, on the basis of i's preferences on issue positions.

The first subset is made up of "dominating" pairs. We say that alternative x dominates alternative x' with respect to i's (separable) preferences -- which we write $x \; D_i \; x'$ -- if and only if $x_h \; P_i \; x'_h$ for all h in K such that $x_h \neq x'_h$ -- in words, if x gives i's preferred position with respect to every issue for which x and x' give different positions. Here is another way to put the matter. Given two alternatives x and x' and voter i's preference ordering, the set K of issues is partitioned into four subsets: K_1, those on which both x and x' give i's more preferred position; K_2, those on which x gives i's more preferred position and x' gives i's less preferred position: K_3, those on which x' gives i's more preferred position and x gives i's less preferred position: and K_4, those on which both x and x' give i's less preferred position. So $x \; D_i \; x'$ if and only if set K_3 is empty.

So the subset of dominating pairs with respect to i's preferences is the set of all pairs of alternatives x and x' such that either $x \; D_i \; x'$ or $x' \; D_i \; x$. The other subset, the set of non-dominating pairs, is simply the complement of the set of dominating pairs.

In is useful, especially when we consider (in the next section) separable majority preference, to subdivide the set of dominating pairs into two subsets according to whether the pairs are "adjacent" or not. Two alternatives are adjacent if they give different positions on just one issue; distinct alternatives are otherwise non-adjacent. (Notice that adjacency is defined with respect to the composition of alternatives, so alternatives are adjacent or not regardless of the preferences of any voter.)

OBSERVATION 7. If alternatives x and x' are adjacent, then either $x \; D_i \; x'$ or $x' \; D_i \; x$ (i.e., x' and x" are a dominating pair) for any voter i.

PROPOSITION 8. (The Adjacency Principle) If alternatives x and x' are adjacent, then for any voter i with separable preferences x P_i x' or x' P_i x according to whether x D_i x' or x' D_i x.

Proof. Immediate from the definition of separability.

PROPOSITION 9. (The Dominance Principle) For any voter i, x D_i x' implies x P_i x'.

Proof. Consider the set K_2 of issues on which x gives i's more preferred position and x' gives i's less preferred position. Consider an alternative z that gives i's more preferred position on some issue in K_2 but is otherwise identical to x'; by the adjacency principle, z P_i x'. If z ≠ x, consider an alternative z' that gives i's more position on some additional issue in K_2 but is otherwise identical to z; by the adjacency principle, z P_i x'. And so forth. We can continue until we have constructed a path of adjacent alternatives such that x P_i ... P_i z" P_i z P_i x'. By the transitivity of individual preference, x P_i x'.

Thus a voter's preferences over dominating pairs of alternatives (whether adjacent or not) are determined by his preferences on issue positions; if we know a voter's issues preferences, we also know which pairs of alternatives are dominating for him and we know his preferences over these pairs of alternatives.

On the other hand, voter i's preferences over non-dominating pairs are not determined by his preferences on issue positions; if we know only a voter's issues preferences, we know which pairs of alternatives are non-dominating for him but we know nothing about preferences over these pairs. Such pairs require the voter to make tradeoffs between preferred positions and thus reflect the voter's intensity of concern over different issues.

However, given separable preferences, there are necessary interdependencies among a voter i's preferences over non-dominating (as well as dominating) pairs of alternatives.

Separability by definition entails interdependencies of the following type. Suppose, for example, that there are three issues, and consider the two non-dominating pairs of alternatives

$(x_1,\overline{x}_2,x_3)$ and $(\overline{x}_1,x_2,x_3)$ and also $(x_1,\overline{x}_2,\overline{x}_3)$ and $(\overline{x}_1,x_2,\overline{x}_3)$. While separability does not determine i's preference between either pair, it does require that i prefer either the first alternative in both pairs or the second alternative in both pairs.

In addition, separability indirectly entails the following interdependency. We say two alternatives x and $\overline{x}$ are opposites if they give different positions on every issue. (Notice that opposite alternatives, like adjacent ones, are defined with respect to the composition of alternatives, not the preferences of any voter.)

PROPOSITION 10. (The Principle of Opposites) Let x and $\overline{x}$ be opposite alternatives, and likewise let x' and $\overline{x}'$ be opposite alternatives. Then, for any voter i with separable preferences, $x\ P_i\ x'$ if and only if $\overline{x}'\ P_i\ \overline{x}$.

Proof. With respect to x and x', let the set K of issues be partitioned into the subsets K_1, K_2, K_3, and K_4 as previously stipulated. With respect to $\overline{x}$ and $\overline{x}'$, let the set K of issues be partitioned into the subsets $\overline{K}_1$, $\overline{K}_2$, $\overline{K}_3$, and $\overline{K}_4$ in parallel manner. Since x and $\overline{x}$ are opposites, and likewise x' and $\overline{x}'$, it follows that $K_1 = \overline{K}_4$, $K_2 = \overline{K}_3$, $K_3 = \overline{K}_2$, and $K_4 = \overline{K}_1$. Suppose $x\ P_i\ x'$; this means i prefers the partial alternative given by x with respect to the issues in $K_2 \cup K_3$ to the partial alternative given by x' with respect to the same issues. That is, with respect to the issues for which x and x' give different positions, x is preferable on some and (in general) x' on others, but on balance i prefers the bundle given by x to the bundle given by x'. Since $\overline{K}_3 \cup \overline{K}_2 = K_2 \cup K_3$ and $\overline{x}$ and $\overline{x}'$ are opposites of x and x' respectively, it follows that voter i also prefers the partial alternative given by $\overline{x}'$ with respect to these issue to the partial alternative given by $\overline{x}$ with respect to the same issues; thus $\overline{x}'\ P_i\ \overline{x}$. By parallel argument, the reverse is also true.

Note that if x and x' are themselves opposites, Proposition 10 reduces to an identity. We may also note that the principle of opposites applies to dominating, as well as non-dominating pairs, but is redundant in the former cases (in that it is implied by the dominance principle).

Repeated application of the principle of opposites gives us the following corollary.

COROLLARY 9.1. If voter i ranks alternative x j-th from the top of his preference ordering, i ranks the opposite alternative $\overline{x}$ j-th from the bottom of his ordering.

Thus the top and bottom halves of any voter's separable preference ordering are, in a sense, "mirror images" of one another.

We can now answer the question posed earlier: if we know a voter preferences on each issue, what can we infer about his preference ordering over all alternatives?

Given a voter preferences on each issue, his most preferred alternative is the alternative that gives his more preferred alternative on every issue (since it dominates every other alternative). By the principle of opposites, his least preferred alternative is the alternative that gives his less preferred alternative on every issue.

The voter second most preferred alternative is some alternative that gives his less preferred position on just one issue h (the issue he "cares about" least); and his second least preferred alternative is the opposite of this. His third most preferred alternative is some alternative that gives his less preferred position on just one other issue j (the issue he "cares about" second least); and his third least preferred alternative is the opposite of this. His fourth most preferred alternative is either some alternative that gives his less preferred alternative on some third issue g (the issue he "cares about" third least) or the alternative that gives his less preferred positions on both of issues h and j; and in either event his third least preferred alternative is the opposite of this. And so forth. In general, the dominance principle establishes a quasi-ordering of preferences over alternative that are related by adjacency. Where this quasi-ordering is incomplete, preferences are completed in a way that reflects how voter i "trades off" different combinations of preferred positions; by the principle of opposites, the way the top half of his ordering is completed determines how the bottom half of his ordering is completed (or vice versa).

In concluding this section, we may note that while separability is defined as a property of preference orderings over alternatives, it implies that voters have induced preferences over individual issue positions. It may in fact be more intuitive to take these issues preferences as the foundation and then build preferences over alternatives from them. In fact, separable preference orderings may be "rationalized" in the following manner. If voter i has separable preferences, it is possible to assign a real number $u_i(x_h)$ to each issue h, which represents i utility for his more preferred position on issue h (where his utility for his less preferred position is taken to be 0), such that $x' \ P_i \ x''$ if and only if $u_i(x') > u_i(x'')$ and $u_i(x) = \Sigma u_i(x_h)$, where the summation is over all issues. Hence, separability (as defined here) is sometimes called additive separability, and the magnitude of the numbers $u_i(x_h)$ reflects how much i "cares about" different issues.[6]

4. Separable Majority Preference

We now consider the structure of majority preference, given that all individual preferences are separable by issues.

PROPOSITION 11. If individual preferences are separable by issues, majority preference is separable by issues.

Proof. Suppose that $(x_A,x_{\bar{A}})$ P $(x'_A,x_{\bar{A}})$ for some x_A and x'_A in X_A and for some $x_{\bar{A}}$ in $X_{\bar{A}}$. We must show that $(x_A,x'_{\bar{A}})$ P $(x'_A,x'_{\bar{A}})$ for any other $x'_{\bar{A}}$ in $X_{\bar{A}}$. By individual separability, $(x_A,x_{\bar{A}})$ P_i $(x'_A,x_{\bar{A}})$ if and only if $(x_A,x'_{\bar{A}})$ P_i $(x'_A,x'_{\bar{A}})$ and conversely for any voter i, so the same is true for any majority of voters.

OBSERVATION 8. Majority preference may be separable by issues even if some individual preferences are not, i.e., the converse of Proposition 11 does not hold.

Given Proposition 11, we can designate the two positions on a given issue X_h as the majority position and minority position, which we henceforth label x_h and $\bar{x}_h$ respectively. Let x^* designate the alternative made up entirely of majority positions and $\bar{x}$ the alternative made up entirely of minority positions.

We now consider which of the properties of separable individual preferences are inherited by separable majority preference. Essentially, this depends on where we previously did or did not invoke the transitive property of individual preference.

As we did in the case of individual preference, we partition all pairs of alternatives into three subsets, now with respect to majority preference: adjacent dominating pairs, nonadjacent dominating pairs, and non-dominating pairs.

We say alternative x dominates alternative x' with respect to (separable) majority preferences -- which we write x D x' -- if and only if x_h P x'_h for all h in K such that $x_h \neq x'_h$ -- in words, if with respect to every issue for which x and x' give different positions, x gives the majority position. The set of dominating pairs of alternatives with respect to majority preference is the set of all pairs x' and x" such that either x' D x" or x" D x'. The set of all non-dominating pairs with respect to majority preference is simply the complement of this.

As before (and with more reason that before), we subdivide the set of dominating pairs into adjacent and nonadjacent subsets.

OBSERVATION 9. If alternatives x and x' are adjacent, then either x D x' or x' D x (i.e., x' and x" are a dominating pair).

PROPOSITION 12. (The Adjacency Principle) If alternatives x and x' are adjacent, then given separable majority preference x P x' or x' P x according to whether x D x' or x' D x.

Proof. The attention of all voters is focused on the single issue with respect to which the alternatives differ and a majority of voters prefer the alternative including the majority position on that issue.

However, while the adjacency principle holds for separable majority preference, the more general dominance principle does not.

PROPOSITION 13. The dominance principle does not hold for majority preference, i.e., x D x' does not imply x P x' (if x and x' are nonadjacent).

Proof. This is because transitivity of P does not hold. Consider Example 3, where x_1 and x_2 are now the majority positions and $\overline{x}_1$ and $\overline{x}_2$ the minority positions. Majority preference relationships (1) - (4) follow from the adjacency principle as they did for individual preference. However, relationship (5) does not follow as it did for individual preference, since transitivity was invoked. Indeed, we can readily construct a preference profile such that the majority and minority positions are as stipulated yet $(\overline{x}_1,\overline{x}_2)$ P (x_1,x_2). This is illustrated by Example 4.

EXAMPLE 4.

P_1	P_2	P_2	Majority Preference (x_1,x_2)	$(x_1,\overline{x}_2)$
(x_1,x_2)	$(\overline{x}_1,x_2)$	$(x_1,\overline{x}_2)$		
$(x_1,\overline{x}_2)$	$(\overline{x}_1,\overline{x}_2)$	$(\overline{x}_1,\overline{x}_2)$		
$(\overline{x}_1,x_2)$	(x_1,x_2)	(x_1,x_2)		
$(\overline{x}_1,\overline{x}_2)$	$(x_1,\overline{x}_2)$	$(\overline{x}_1,x_2)$	$(\overline{x}_1,\overline{x}_2)$	$(\overline{x}_1,x_2)$

While the pair (x_1,x_2) and $(\overline{x}_1,\overline{x}_2)$ is dominating with respect to majority (and voter 1's) preferences, it is not dominating with respect to voter 2's and 3's preferences; voters 2 and 3 have opposite preferences with respect to issue positions but happen to have similar preferences with respect to the pair (x_1,x_2) and $(\overline{x}_1,\overline{x}_2)$. Since together they constitute a majority, $(\overline{x}_1,\overline{x}_2)$ P (x_1,x_2).

While the majority preference relationships implied by the adjacency principle are acyclic, they are also intransitive, so cyclical majority preference can -- though need not -- occur with respect to dominating pairs of alternatives.

Like individual preference, majority preference with respect to non-dominating pairs of alternatives is not determined by issue preferences (i.e., the identification of majority and minority issue positions). But the same interdependencies apply to majority preference. First, the interdependency entailed by the definition of separability of course applies, and the derivative principle of opposite applies as well.

PROPOSITION 14. (The Principle of Opposites) Let x and $\overline{x}$ be opposite alternatives, and likewise let x' and $\overline{x}'$ be opposite alternatives. Then, if majority preference is separable, x P x' if and only if $\overline{x}'$ P $\overline{x}$.

Proof. By Proposition 10, any voter prefers x to x' if and only if he prefers $\overline{x}'$ to $\overline{x}$. So a majority of voter prefer x to x' if and only if the same majority prefers $\overline{x}'$ to $\overline{x}$.

Majority preference on pairs of non-dominating alternatives may form a cycle, but the principle of opposite requires that the reverse cycle is formed over the opposite pairs.

The following is a version of a theorem given in Kadane (1972), Miller (1975, 1977), and elsewhere.

PROPOSITION 15. Given separable majority preference, the alternative x^* that includes the majority position on every issue belongs to the top cycle of X.

Proof. Consider any alternative x' other than x^*. Replace any minority position in x' with the majority position on the same issue to get alternative x"; then, by the adjacency principle, x" P x'. If x" ≠ x^*, repeat; and so forth until we reach x^*. Thus there is a P-path from x^* to x', i.e., to any other alternative. In a tournament, a alternative x belongs to the top cycle if and only if there is a P-path from x to each other alternative (Miller, 1977).

COROLLARY 15.1. If there is a Condorcet alternative in X, it is x^*.

Of course, Example 4 shows that the converse of Corollary 15.1 does not hold.

OBSERVATION 9. The alternative x^* that includes the majority position on every issue in X may not be a Condorcet alternative

5. The Effectiveness of a Coalition of Minorities

Example 4 illustrates the phenomenon of a "coalition of minorities," first explicitly identified and named by Downs (1957, pp. 55ff). The majority preference relationship $(\overline{x}_1, \overline{x}_2)$ P (x_1,x_2) which, in conjunction with other relationships determined by the adjacency principle, creates the cycle is effected by the coalition {2,3}. This comes about because each of the following conditions holds:

(1) each of voters 2 and 3 prefers the majority position on one issue and the minority position on the other; but

(2) they prefer the majority and minority position on different issues; and

(3) each prefers to get his way on the issue on which he favors the minority position; and

(4) together 2 and 3 constitute a majority.

It is such coalitions of minorities that can bring about violations of the dominance principle with respect to majority preference over nonadjacent alternatives. In fact we can define an <u>effective coalition of minorities</u> formally as a coalition that brings about a majority preference relationship x P x' for some pair of alternatives x and x' such that x' D x.

In Example 4, the coalition of minorities {2,3} is effective against x^*, the alternative that gives the majority position on every issue. But if the total number of issues is three or more, a coalition of minorities may be effective against other alternatives as well. However, a coalition of minorities that is effective against a dominating alternative other than x^* is also effective against x^*.

PROPOSITION 16. Given separable preferences over three or more issues, if there is some pair of alternatives x' and x" such that x' ≠ x', x' D x", and x" P x' with the effective coalition (of minorities) C, then there is some alternative x such that x P x^* with the same effective coalition (of minorities) C.

<u>Proof</u>. Let X_A be the set of issues with respect to which x" and x' differ, x' giving majority

positions and x" giving minority positions on all issues in A; call these partial alternatives x'_A and $x''_{\bar{A}}$ respectively. Since $x' \neq x^*$, x' must contain at least one minority position, which implies that $A \neq K$, i.e., these are strict partial alternatives with some non-empty complement $x_{\bar{A}}$. Now consider the complement $x^*_{\bar{A}}$ made up entirely of majority positions. Since by assumption $x'' = (x''_A, x_{\bar{A}})$ Pi $(x'_A, x_{\bar{A}}) = x'$ for all i in C, by separability $(x''_A, x^*_{\bar{A}})$ P_i $(x'_A, x^*_{\bar{A}})$ for all i in C. But $(x'_A, x^*_{\bar{A}}) = x^*$, so there is some $x = (x''_A, x^*_{\bar{A}})$ such that x P x^* with the same effective coalition C.[7]

Thus conditions that render coalitions of minorities ineffective, while they do not preclude all cycles in separable majority preference, do preclude cycles including x^* and thus preclude top cycles. Given Proposition 16, we can focus exclusively on coalitions of minorities that are effective against x^*. We want to identify the family of subsets of (two or more) issues with respect to which some coalition of minorities may be effective.

Each issue X_h partitions the set N of voters into two complementary subsets according the their preferences on that issue. Let N_h designate the set of voters, n_h in number, who prefer the majority position on issue h; and let $\overline{N}_h$ designate the set of voters, $\overline{n}_h$ in number, who prefer the minority position on issue h. By assumption, $n_h + \overline{n}_h = n$ and, by definition, $\overline{n}_h < n/2 < n_h$ for all h.

Let $p_h = n_h/n$ and $\overline{p}_h = \overline{n}_h/n$ designate the proportion of all voters who prefer the majority and minority positions respectively on issue h. We call p_h the <u>level of popularity</u> of x_h. To the extent that p_h approaches .5, we call issue h <u>consensual</u>; to the extent that p_h approaches 1, we call issue h <u>divisive</u>.

Consider the set of issues {1,2}. We examine the conditions under which the alternative $(\overline{x}_1, \overline{x}_2, x_3, \ldots, x_k)$ is majority preferred to $(x_1, x_2, x_3, \ldots, x_k)$? Since all preferences are separable, we can ignore the fixed complement $(x_3, \ldots, x_k)$ (which, by Proposition 16, could just as well be any other fixed complement) and refer simply to (x_1, x_2) and $(\overline{x}_1, \overline{x}_2)$.)

We can partition the set of voters into four subsets or <u>clusters</u> according to their (separable) preferences regarding issues 1 and 2, as follows. The <u>majority cluster</u> (or "pivot," in Lin's (1991) terminology) is the intersection $N_1 \cap N_2$, composed of those voters who prefer the majority position on both issues; by the dominance principle, all such voters prefer (x_1,x_2) to $(\overline{x}_1,\overline{x}_2)$. The minority cluster is the intersection $\overline{N}_1 \cap \overline{N}_2$, composed of those voters who prefer the minority positions on both issues; by the dominance principle, all such voters prefer $(\overline{x}_1,\overline{x}_2)$ to (x_1,x_2). The mixed clusters are the two intersections $N_1 \cap \overline{N}_2$ and $\overline{N}_1 \cap N_2$, composed of those voters who prefer the majority position on one issue and the minority position on the other. Their preferences are not governed by the dominance principle; rather their preferences reflect the tradeoffs such voters make between issues 1 and 2. Since voters with the such issue preferences may make different tradeoffs, different voters in the two mixed clusters, as well as within the same mixed cluster, may have different preferences between (x_1,x_2) and $(\overline{x}_1,\overline{x}_2)$.

With respect to this pair of issues, a coalition of minorities is a superset of -- and therefore cannot be smaller than -- the minority cluster; at the same time it is a subset of -- and therefore cannot be larger than -- the union of the minority cluster and the mixed clusters. Within these bounds, the size of a coalition of minorities depends on how many voters in the mixed clusters have the appropriate preference, i.e., for $(\overline{x}_1,\overline{x}_2)$ over (x_1,x_2). Whether a coalition of minorities can be effective for the majority preference relationship $(\overline{x}_1,\overline{x}_2)$ P (x_1,x_2) depends, therefore, on the size of the clusters and the distribution of intensities within the mixed clusters. Let us examine these matters in more detail.

The size of the clusters depends in turn on two other factors: the level of popularity of each majority position and the degree of association or reinforcement between preferences on the two issues.

Each level of popularity is indicated by p_1 and p_2, the proportion of voters who prefer the majority position on issue 1 and issue 2 respectively.

The degree of association or positive reinforcement between preferences on the two issues may be indicated by the parameter r, which represents, as a proportion of all voters, the upward deviation in the size of each of the majority and minority clusters, and the complementary downward deviation of each of the mixed clusters, from the "baseline" proportions that would result if preferences on the two issues were statistically independent.[8] In general, then, the frequency distribution of voters over the four clusters is as shown below.

Preferred Position

	x_2	$\bar{x}_2$	
x_1	Majority Cluster $p_1p_2 + r$	Mixed Cluster $p_1\bar{p}_2 - r$	p_1
$\bar{x}_1$	Mixed Cluster $\bar{p}_1p_2 - r$	Minority Cluster $\bar{p}_1\bar{p}_2 + r$	$\bar{p}_1$
	p_2	$\bar{p}_2$	1

If preferences on the two issues are not associated or, equivalently, are perfectly <u>crosscutting</u> (so that people who favor the majority position on one issue are neither more nor less likely than others to favor the majority position on the other issue), r equals zero, and the proportion of voters in the majority cluster is p_1p_2, in the minority cluster is $\bar{p}_1\bar{p}_2$, and in the two mixed clusters is $p_1\bar{p}_2$ and $\bar{p}_1p_2$. (These are the "baselines.")

To the extent that issue preferences are positively associated or, equivalently, are <u>positively reinforcing</u> (so that people who favor the majority position on one issue are more likely than others to favor the majority position on the other issue), r is positive, so the majority and minority clusters each deviate upwards from the baseline by r and the two mixed clusters each deviate downwards by r. When preferences are as positively reinforcing as possible (given the popularity levels p_1 and p_2), at least one mixed cluster is empty; since no cluster size can be negative, the maximum value of r, given p_1 and p_2, is the smaller of $p_1\bar{p}_2$ and $\bar{p}_1p_2$. In turn, the maximum value of either $p_1\bar{p}_2$ or $\bar{p}_1p_2$ occurs when both issues are as divisive as possible (that is, have the lowest possible level of popularity), i.e., when $p_1 = p_2 \approx .5$, so $p_1\bar{p}_2 = \bar{p}_1p_2 \approx .25$.

To the extent that issue preferences are negatively associated or, equivalently, are negatively reinforcing (so that people who favor the majority position on one issue are less likely than others to favor the minority position on the other issue), r is negative, so the majority and minority clusters each deviate downwards from the baseline by r and the mixed clusters each deviate upwards by r. When preferences are as negatively reinforcing as possible (given the popularity levels p_1 and p_2), the minority cluster is empty; thus the minimum value of r, given p_1 and p_2, is $-\overline{p}_1\overline{p}_2$ and the maximum value of $\overline{p}_1\overline{p}_2$ occurs when both issues are as divisive as possible, i.e., when $p_1 = p_2 \approx .5$ and $\overline{p}_1\overline{p}_2 \approx .25$.

It is useful the summarize these observations formally.

OBSERVATION 10. The parameter r indicating degree of reinforcement has these bounds:

$$-.25 < -\overline{p}_1\overline{p}_2 \leq r \leq \min[p_1\overline{p}_2, \overline{p}_1 p_2] < .25.$$

PROPOSITION 17. Given just two issues, the majority cluster is always larger than the minority cluster and is never empty.

Proof. Since $p_h > \overline{p}_h \geq 0$ for all h, $p_1 p_2 + r > \overline{p}_1\overline{p}_2 + r \geq 0$.

The other determinant of an effective coalition of minorities, in addition to cluster size, is the distribution of intensities within the mixed clusters. Let F_{maj} (the fraction with majority intensity) be the fraction of voters in the mixed clusters who prefer the alternative that gives both majority positions to the alternative that gives both minority positions -- that is, who prefer getting their way only on the issue on which they prefer the majority position to getting their way only on the issue on which they prefer the minority position. Let F_{min} (the fraction with minority intensity) be the fraction of voters in the mixed clusters who prefer the alternative that gives both minority positions to the alternative that gives both majority positions -- that is, who prefer getting their way only on the issue on which they prefer the minority position to getting their way only on the issue on which they prefer the majority position. Since we continue to assume strong preferences, $F_{maj} + F_{min} = 1$.[9]

Again it is useful to summarize these observations formally.

OBSERVATION 11. The condition for an effective coalition of minorities is:

$$(\overline{p}_1\overline{p}_2 + r) + F_{min}(p_1\overline{p}_2 + \overline{p}_1 p_2 - 2r) > .5 >$$
$$(p_1 p_2 + r) + F_{maj}(p_1\overline{p}_2 + \overline{p}_1 p_2 - 2r).$$

In words, the minority cluster plus the fraction of the mixed cluster with minority intensity must be greater than one half which in turn is greater than the majority cluster plus the fraction of the mixed clusters with majority intensity.

We can readily identify two necessary conditions for an effective coalition of minorities.

PROPOSITION 18. A coalition of minorities is effective only if:

(a) $p_1 p_2 + r < n/2$, i.e., the majority cluster is of less than majority size; and

(b) $F_{min} > .5$, i.e., the fraction of the mixed clusters with minority intensity is greater than one half.

Proof. Part (a) is immediate from Observation 11; part (b) is immediate from Observation 11 in conjunction with Proposition 17.

Turning the proposition around, one condition to preclude an effective coalition of minorities is that the majority cluster itself be of majority size; this can result if one or both issues are quite consensual or if preferences are quite positively reinforcing.

Otherwise, the effectiveness of a coalition of minorities depends on the preferences of voters in the mixed cluster. If F_{maj} is at least as great as F_{min}, a coalition of minorities cannot be effective.[10] (In this sense, an "impartial" distribution of issue intensities means that coalitions of minorities can rarely be effective.) A coalition of minorities can be effective only if F_{min} is greater than F_{maj}, i.e., if most voters who are in the majority on one issue and in the minority on the other issue care more about the latter and thus prefer $(\overline{x}_1, \overline{x}_2)$ to (x_1,x_2). The required advantage of F_{min} over F_{maj} depends in turn on the size of the majority and minority clusters.

These relationships are depicted graphically in Figures 1 and 2 for cases in which the two issues are equally divisive (or consensual), i.e., in which $p_1 = p_2$.[11]

Figure 1 shows the "space" of all combinations of popularity p (equal for the two issues) and reinforcement r. The area labelled "infeasible" represents combinations that lie outside of the bounds specified in Observation 10. When $p \approx .5$, the range of r is maximum, from -.25 to +.25. As p increases, the range of r becomes more restricted -- at first mostly from below -- until p = 1, at which the range of r converges on zero. Within the space of feasible combinations, the diagonally hatched area represents combinations of p and r that entail a majority cluster of majority size, precluding an effective coalition of minorities regardless of the magnitude of F_{min}. The remaining (white) area of the feasible space represents combinations of p and r that permit an effective coalition of majorities contingent upon an adequately large F_{min}. The labelled contour lines, all emanating from of the upper left corner (p = .5, r = +.25) of the space indicate the required level of majority intensity. For example, $F_{min} = .75$ will entail an effective coalition of minorities for any combination of p and r in the feasible area of the space inside (to the left of) the line labelled $F_{min} = .75$.

Figure 2, by holding popularity constant at .6 (for both issues), may make the relationship between degree of reinforcement and the magnitude of F_{min} necessary for an effective coalition of minorities more apparent. Figure 2 is in effect a two-dimensional representation of the information on the vertical line p = .6 in Figure 1. The vertical dimension in Figure 2, like Figure 1, corresponds to values of r (but here restricted to its feasible range from -.16 to +.24) and also to size of the majority cluster (which, with p fixed, depends only on r) and to the measure XC of "cross-cuttingness" (see footnote 8). The horizontal dimension represents values of F_{min} (or F_{maj}). Points in the resulting rectangular space belong to the cross-hatched or white region according to whether or not a coalition of minorities is effective at that combination of reinforcement and intensity. The curved line separating the two regions in effect is a plot of the F_{min} contours as they intersect the line p = .6 in Figure 1. Figure 2 shows graphically the two conditions identified in Proposition 18 as sufficient to preclude an effective coalition of minorities: (a) majority cluster of majority size (called a "universal majority" in the next section) and (b) a generalized passionate majority, i.e., $F_{maj} \geq .5$ (see footnote 11).

By exactly similar analysis, we can check whether a coalition of minorities can be effective with respect to any other pair of issues in X. And even if no coalition of minorities can be effective on pair of issues (or in any event), such a coalition may be effective with regard to some larger set of issues. The general nature of the previous analysis may be extended to consideration of this possibility. As before, there is a single majority cluster (that prefers the majority position on every issue) and a single minority cluster (that prefers the minority position on every issue), but there are now many (in general, $2^h - 2$, where h is the number of issues in the set) mixed clusters, and -- unless preferences are highly (and positively) reinforcing -- almost all voters will belong to mixed clusters. (Also, once the number of issues exceeds two, the majority cluster may be smaller than the majority cluster and may be empty.) As we consider the effectiveness of coalitions of minorities with respect an expanding set of issues, there are two competing considerations. On the one hand, the potential coalition size increases (unless preferences are perfectly reinforcing), as more and more voters are likely to find themselves in the minority on at least one issue. On the other hand, finding the right distribution of intensities becomes more complex. Voters in some mixed clusters prefer the majority position most issues in the set, and it is unlikely that they will care enough about the smaller number of other issues to prefer to get their way on those at the cost of not getting their ways on the greater number of issues on which they prefer the majority position. On the other hand, voters in other mixed clusters prefer the minority position on most issues in the set, it is likely that they will care enough about these issues to prefer to get their way on them at the cost of not getting their ways on the small number of issues on which they prefer the majority position. Finally, other voters are in "balanced" mixed clusters who prefer the majority positions on about half the issues and the minority positions on the other half; such voters face tradeoffs similar to voters in the two mixed clusters in the two-issue case. However, to the extent that issues are consensual more than divisive (to the extent that p exceeds .5) on average, the first class of mixed clusters will be the largest, making an effective coalition of minorities with respect to the expanded set of issues rather unlikely. On the other hand, as the set of issues expands, it evidently becomes more likely that a coalition of minorities will be effective with respect to <u>some</u> subset of issues.[12]

5. The Separate Issues Model

The discussion in Miller (1983) was (informally) conducted within the framework of multiple issues with separable preferences.

<u>The Separate Issues Model</u>

(a) a set of k dichotomous issues; and consequently

(b) a set X of 2^k alternatives;

(b) strong <u>separable</u> voter preference orderings over the alternatives in X; and

(c) the number of individuals is odd; so

(d) majority rule is strong (a tournament).

In the separate issues framework, we call a voter who prefers the majority position to the minority position on every issue X_1 through X_k, i.e., who is a member of the (universal) majority cluster, a <u>universal winner</u>. A <u>universal loser</u> is a voter who prefers the minority position to the majority on every issue, i.e., who is a member of the (universal) minority cluster. A <u>universal majority</u> is a set of voters of majority size all of whom are universal winners, i.e., a (universal) majority cluster of majority size.

OBSERVATION 12. All universal winners have identical preferences over issue positions, but their preferences over (non-dominating pairs of) alternatives may differ.

OBSERVATION 13. All universal losers have identical preferences over issue positions, but their preferences over (non-dominating pairs of) alternatives may differ.

OBSERVATION 14. Universal winners and losers have directly opposed preferences over issue positions, but their preferences over (non-dominating pairs of) alternatives may not be directly opposed.

Permanent winners, losers, and majorities may be defined as before, i.e., in terms of preferences over all alternatives, not just issue positions. We may then ask what is the relationship between permanent winners, losers, and majorities and universal winners, losers and majorities. The answer is apparent.

PROPOSITION 19. Given separable preferences, permanent winners are also universal winners but not conversely; permanent losers are also a universal losers but not conversely.

Proof. A permanent winner wins on every pair of alternatives and by Proposition 1 there is a majority ordering and his most preferred alternative is at the top of this ordering. By Corollary 15.1, this is the alternative x^* that includes the majority position on every issue. Thus the permanent winner must prefer the majority position on every issue and is a universal winner. In like manner, a permanent loser loses on every pair of alternatives. Hence by Proposition 1 there is a majority ordering and his most preferred alternative is at the bottom of this ordering. The alternative x^* that includes the majority position on every issue is at the top of this ordering so, by the principle of opposite applied to majority preference, the alternative at the bottom is $\overline{x}$, giving the minority position on every issue. Thus the permanent loser must prefer the minority position on every issue and is a universal loser.

PROPOSITION 4. Given an arbitrary separable preference profile, we may find

(a) both a universal winner and a universal loser; or

(b) a universal winner but no universal loser; or

(c) a universal loser but no universal winner (if there are three or more issues); or

(d) neither a universal winner nor a universal loser.

Proof. It is sufficent to provide an example for each case (a) -(d).

(a) This occus in any two-issue case with a non-empty minority cluster.

(b) This occurs in any two-issue case with an empty minority cluster (maximum negative reinforcement).

(c) This is demonstrated by the following example (only first preferences are shown).

EXAMPLE 5.

2 voters	2 voters	2 voters	1 voter
$(x_1,x_2,\overline{x}_3)$	$(x_1,\overline{x}_2,x_3)$	$(\overline{x}_1,x_2,x_3)$	$(\overline{x}_1,\overline{x}_2,\overline{x}_3)$

(d) This is would occur if the voter with first preference $(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ were removed from Example 5.

We may note in conclusion that, since by definition $p_h > \overline{p}_h$ for all h, universal (and near-universal) winners are a priori more common than universal (and near-universal) losers. Moreover, although the number of universal (and near-universal) winners and of universal (and near-universal) losers both tend to diminish as the number of issues expands (unless there is perfect positive reinforcement), the relative advantage of winners over losers tends increase (especially if preferences are crosscutting); see for example the illustrative numbers in Table 2 below.

6. Pluralism and Social Choice

The substantive argument in Miller (1983) was founded on two formal propositions which were, however, only informally stated and justified.

1. The more issues there are, and the more preferences on these issues are crosscutting, the more equally political satisfaction with outcomes will be distributed (which promotes political stability).

2. The more issues there are, and the more preferences on these issues are crosscutting, the more likely it is that some coalition of minorities will be effective and consequently that majority preference over alternatives will be cyclical with the result that there will be no "stable" Condorcet winner (which promotes political stability by inducing current loser to keep on "playing the game").

With respect to the first proposition, Feld and Grofman make the point that Miller may seem to want to preclude universal (or near-universal) winner by suggesting that doing so is necessary and/or sufficient to precluding universal (or near-universal) losers (which is the real desideratum, in terms of political stability). Whether or not this was suggested, the implicit proposition is clearly wrong, as indicated by Proposition 20. But more generally, it could surely be argued that a (more or less) equal distribution of political satisfaction does not promote political stability unless that (more or less) equal distribution of political satisfaction is also at least moderately high.

In any event the first underlying formal proposition identified above seems to be justified. Let us use the following simple (inter-personally comparable) scale of satisfaction: a voter's levels <u>level of satisfaction</u> is equal to the number of issue on which he prefers the majority positions. Now let us examine the simple two-issue case in this respect.

Frequency	Level of Satisfaction	Deviation from Mean
$p_1p_2 + r$	1.0	$1 - .5(p_1 + p_2)$
$p_1\overline{p}_2 - r$	0.5	$.5(p_1 + p_2) - .5$

$\overline{p}_1 p_2 - r$	0.5	$.5(p_1 + p_2) - .5$
$\overline{p}_1 \overline{p}_2 + r$	0.0	$.5(p_1 + p_2)$

If we multiply each frequency by the corresponding level of satisfaction and add up the products, we get the average (mean) level of satisfaction, which is $.5(p_1 + p_2)$. The significant point is that the parameter r disappears from the expression -- that is, the mean level of satisfaction depends only on how consensual or divisive the issues are and not at all on how reinforcing or crosscutting preferences are. Indeed, the conclusion can be reach directly and intuitively by observing that the individual level of satisfaction on any single issue X_k is either 1 or 0 and that the average level of satisfaction is simply the proportion who are satisfied, i.e., p_k. Thus the average level of satisfaction on the two issues is simply the average of p_1 and p2. From this it is clear that the result generalizes to any number k of issues, i.e., the average level of satisfaction is simply $\Sigma p_h/k$, where the summation is over all h in K.

Our concern, however, is with the distribution of satisfaction and how equal or unequal this is. Let us measure the inequality in the distribution of satisfaction in terms of the average (absolute) deviation from the mean level of satisfaction (as shown in the third column above for the two- issue case).

PROPOSITION 21. In the two-issue case, inequality in the distribution of satisfaction is equal to

$$2(p_1p_2 + r)[1 - .5(p_1 + p_2)],$$

that is,

$$2 \times (\text{size of majority cluster}) \times (1 - \text{mean satisfaction}).$$

Proof. Straightforward (if tedious) algebraic derivation from entries in table above.

Thus, while the average level of satisfaction is independent of the degree of reinforcement of preferences, inequality in the distribution of satisfaction increases as reinforcement increases. The latter also depends on the popularity of the majority positions, but greater popularity has both a positive effect (by increasing the size of the majority cluster) and a negative effect (by increasing mean satisfaction), so the two effects tend to balance out.

Table 1 displays inequality of satisfaction for selected levels of popularity (equal for both issues, as in Figure 1, to which Table 1 may be compared). We may observe that inequality of satisfaction does depend on both popularity and reinforcement, but it is a fair generalization to say that reinforcement is more important, especially when issues are quite divisive. Looking down the parallel columns, we see that, when issues are maximally divisive, each unit increase in reinforcement produces a unit increase in inequality. As issues become more consensual, a given difference in reinforcement make less difference for inequality (and the range of possible reinforcement becomes more restricted). Looking across rows, we see a curvilinear relationship between popularity and inequality, but variation in popularity makes relatively little difference except at very high levels among somewhat reinforcing issues.

Increasing the number of issues (while keeping average satisfaction constant) also reduces inequality in satisfaction to the extent that preferences across issues are crosscutting. This is illustrated in Table 2, for cases in which p = .6 for all issues; the relative frequency of each level of satisfaction is shown depending on whether preferences are perfectly reinforcing, perfectly crosscutting, or intermediate (where frequencies simply the midpoints between the two extremes). Inequality (mean deviation) in satisfaction is shown at the bottom of each frequency distribution. For reinforcing preferences, level of satisfaction remains polarized regardless of the number of issues. For crosscutting preferences, level of satisfaction becomes more closely bunched around the mean level (i.e., p) as the number of issues increases.[13]

In general then, the first formal proposition from Miller (1983) seems to be sustained. The second is sustained by the general thrust of the analysis in Section 5, with an important proviso suggested by the analysis in Lin (1991). Lin demonstrates, in the two-issue case, that less crosscutting preferences can increase the probability of an effective coalition of minorities (in effect, by reducing the required fraction with minority intensity). He also demonstrated that the size of the majority cluster -- which he call "the pivot" -- is more reliably related to the effectiveness of a coalition of minorities than degree of crosscuttingness in preferences. The apparent discrepancy between Miller's argument and Lin's analysis can be accounted for in the following way.

The original distinction in sociological literature between "reinforcing" and "crosscutting" patterns of cleavage (to which Miller referred) concerned cleavages that had no common polarity such as "low" and "high" (for example, religious affiliation, language group, region, occupation, etc.). In this context, one could only talk about pairs of cleavages being either (in some degree) reinforcing (or associated) or crosscutting (or unassociated) -- no distinction between positive and negative reinforcement (or association) was appropriate. (The Rae-Taylor measure XC of "cross-cutting," which Lin (1991) uses, explicitly reflects this approach; see footnote 8.)

However, when we extend the notion of crosscutting versus reinforcement to preferences on dichotomous issues, in conjunction with a focus on the operation of majority rule which leads us to label issue positions as "majority" versus "minority," a common polarity is introduced. Thus it now it possible and natural to distinguish between positive versus negative reinforcement, as was done on pp. 32 and 33 of this paper. While it was not totally explicit, the discussion of "reinforcing preferences" in Miller (1983) referred to positively reinforcing preferences only -- for example, "reinforcing divisions of a population into majority and minority groups" (p. 740). (The possibility of negatively reinforcing preferences was not explicitly taken account of.)

The analysis in section 5 (for example, as summarized in Figure 2) shows that the probability of an effective coalition of minorities increases (i.e., the required F_{min} decreases) monotonically as the degree of positive reinforcement decreases towards crosscuttingness (popularity remaining constant). But the probability of an effective coalition of minorities is not at its maximum (the required f_{min} is not at its minimum) when preferences are perfectly crosscutting. The probability continues to increase (the required F_{min} continues to decrease) -- though only slightly -- as preferences become negatively reinforcing, and the maximum (minimum) is achieved at maximum negative (or minimum positive) reinforcement. Lin (1991), using the Rae-Taylor XC measure, in effect "folds over" the relationship between the probability of an effective coalition of minorities and the degree of reinforcement at the $r = 0$ ($XC = .5$) point, therby conflating positive and negative reinforcement and producing what he calls the "bifurcation effect." On the other hand, "pivot" (majority cluster) size is a direct function of degree of positive reinforcement (as well as popularity), so Lin finds a clear relationship between this and the probability of an effective coalition of minorities.

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FOOTNOTES

1. Note that, while preferences in Example 1 are single-peaked on every triple of alternatives (and therefore there is a majority ordering), they are not single-peaked on X as a whole.

2. This result contradicts an observation made in passing by Feld and Grofman (p. 5), who probably implicitly assume <u>symmetric</u> single-peaked preferences. Se Proposition 7 below.

3. Such preferences are also called (one-dimensional) <u>Euclidean</u>. It might seem that we could define symmetric single-peaked (or Euclidean) preferences more straightforwardly as follows: voters with the same first preference have identical preference orderings. It is true that, given symmetric single-peaked preferences, voter with the same <u>ideal point</u> have identical (ordinal) preferences. But, given a finite set X of alternatives (points along the continuum), voters with different ideal points (and thus different preferences) may share the same most preferred alternative in X. (Thus the set of voters who are median with respect to ideal points along the continuum are typically a proper subset of those who are median with respect to X; indeed, typically the first is a one-element set.) And it is also true the even if all voters with the same ideal point have identical (single-peaked) preferences, their preference may not be symmetric. (They may share the same asymmetry in their utility functions.)

4. The permanent winner is the voter who is median, not merely with respect to the alternatives in X, but also with respect to ideal points along the continuum in which the elements of X are embedded. Some other median (with respect to X) voters may have identical preferences over X, in which case they also are permanent winners in X. (Recall footnote 3.)

5. Where alternatives are generated by issues in this fashion, it may be clarifying to refer to them as "platforms," as a platform may be thought of as giving a position on every issue.

6. Given that we assume each voter i's preferences are strong, we must be care to select the utility numbers so that no "ties" can occur.

7. The converse is true provided A has a non-empty complement.

8. This deviation from statistical independence can be specified by the single parameter r in the special case in

which we are considering just two issues, both of which are dichotomous, with fixed levels of popularity. Put otherwise, a 2×2 contingency table with known marginals has just one degree of freedom -- once a single cell value is fixed, the other three are determined.

This measure is rather different from the measure of "cross-cutting" XC proposed by Rae and Taylor (1970) and employed ny Lin (1991). First, XC reflects "fragmentation" (popularity) on each issue taken alone, as well as the association between them, whereas r explicitly measures the deviation from the baseline implied by the given (marginal) popularity levels assuming no association. Second, XC makes no distinction between positive and negative reinforcement -- it is, in effect, "folded over" at the point of perfect crosscuttingness. There are other more technical differences as well.

9. Of course, the two fractions F_{maj} and F_{min} need not be the same in both the two mixed clusters, but all that matters in determining the effectiveness of a coalition of minorities is the overall fraction.

10. Downs (1957) refers to the ineffectiveness of a coalition of minorities as <u>rule of the passionate majority</u>. The terminology may be slightly misleading. First, no interpersonal comparisons of preference intensity are relevant -- the question is how <u>individual voters</u> make issue tradeoffs within their own preference orderings. Secondly, how voters within either the majority or minority clusters make such tradeoffs is also irrelevant -- the question is only how voters within the mixed clusters make such tradeoffs. Finally, it is not necessary for all voters in the mixed cluster to prefer to get their way on the issue on which they prefer the majority position in order to produce "rule of a passionate majority"; it is always sufficient that at least one-half of such voters have such preferences (we refer to this condition as a "generalized passionate majority"), but even this is almost never necessary. How great this proportion must be depends on the size of the several clusters.

11. Figures 1 and 2 are drawn on the assumption that the number of voters n is infinite (or, in any case, very large); if n were small, the lines indicating relationships would move in discrete jumps, rather than continuous curves.

12. Note that even when we have identified all subsets of (two or more) issues with respect to which a coalition of minorities is effective against x^*, we still have not determined the scope of the top cycle set. For example, suppose that the majority alternative x^* (which we know

belongs to the top cycle set) is beaten by just one alternative. One or more alternatives containing a different pair of minority positions or three or more minority positions may yet be in the top cycle set, since such an alternative may beat (say) an alternative containing one minority position which beats an alternative containing two majority positions which beats x^*.

In general, if a given alternative x' beats x^* (and is therefore in the top cycle), at a minimum every other alternative x" such that x" D x' is also in the top cycle. (This implies that if an alternative containing h minority positions is in the top cycle, there is at least one alternatives containing h-1, h-2,...,1 minority positions that also are in the top cycle). But, even if no alternative other than x' beats x^*, other alternatives (that do not dominated x') may belong to the top cycle as well, depending on the nature of majority preference relationships not determined by the adjacency principle. In particular, while it is true that a single coalition of minorities effective against x^* implies (given a sufficient number of issues) an "epidemic" of parallel relationships by virtue of Proposition 16 (or, more precisely, by its partial converse; see footnote 7) and while these relationships -- in conjunction with other majority preference relationships entailed by the adjacency principle -- necessarily result in cycles, such cycles need not intersect, and therefore need not be part of, the top cycle. Put concisely, there is no "global cycling theorem" for separable preferences.

A further and still more difficult analytic problem is to say something about the uncovered set (Miller, 1980) or the Banks set (Banks, 1985; Miller et al., 1990) under separable preferences.

13. If a few pairs of issues were actually negatively reinforcing, universal (or even near-universal) winners and losers could be totally eliminated, further reducing inequality in satisfaction. Note, however, that preferences cannot be consistently negatively reinforcing, since if issues 1 and 2 are negatively reinforcing, and issues 2 and 3 are negatively reinforcing, then issues 1 and 3 are positively reinforcing.