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Logrolling and The Arrow Paradox: A Note

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In an article that appeared in a recent issue of this journal, Peter Bernholz shows that "logrolling implies the presence of the Arrow-paradox at least for two issues with two alternatives."[1] In this note I will present an alternate proof of this point, which is simple and direct and covers any number of issues with any number of alternatives. I will also point out that this argument has been made before – indeed, it has previously appeared in this journal.

Consider several issues, A, B, etc., each including several alternatives; e.g., a_1, a_2, etc., for issue A. Each distinct combination of alternatives, one from each issue, specifies a "platform," or "policy package," or (to use Bernholz's term) a "complex alternative;" I shall use the term "platform."

Consider also an odd number N of voters with strong preferences over the platforms. I assume (as does Bernholz) that voters' preferences are "separable" or "independent" with respect to the several issues – that is, each voter can establish a preference ordering of the alternatives in any one issue considered alone. For example, with respect to issue A, voter i prefers a_1 to a_2 regardless of how other issues may be resolved; put otherwise, he prefers any platform including a_1 to the otherwise identical platform including a_2. (This assumption certainly is not

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[1] Logrolling, Arrow Paradox and Cyclical Majorities," *Public Choice*, Vol. 15 (Summer, 1973), p. 88.

plausible in every circumstance, but it does appear to be a necessary one in considering the phenomenon of logrolling as it is typically formulated. It should be noted that this assumption implies a definite restriction on admissible individual preference orderings over platforms.)

One platform must be selected by means of voting. I will restrict my analysis to simple majority voting as normally employed in parliamentary bodies.

It is convenient initially to consider the case in which every issue is dichotomous, i.e., includes exactly two alternatives, which we may then designate the "majority alternative" and the "minority alternative" according to the preferences of voters with respect to that issue. There is one platform, call it P★, that includes the majority alternative in every issue; if the issues were voted on sequentially without any cooperation among voters, P★ obviously would be the voting decision. Once we admit the possibility of full cooperation among voters (i.e., of logrolling agreements, which may require voters to vote contrary to their preferences on some issues), the question is whether any coalition of voters is "effective" against P★, i.e., whether there is any coalition of a majority of voters all of whom prefer some other platform, say P_o (necessarily including at least one minority alternative), to P★ – in Bernholz's terms, whether a "logrolling situation" exists. Since it includes a majority of voters, the members of such a coalition, often characterized as a "coalition of minorities," can of course assure (by appropriately coordinating their voting choices) that P_o becomes the voting decision.

Suppose that P★ is "dominated" in this sense–in other words, suppose that a "logrolling situation" exists. What Bernholz demonstrates, for the two issue case, is that an "Arrow paradox" then exists; i.e., that majority preference over platforms is cyclical and every platform is dominated by some other platform. Bernholz's proof, which focuses on sets of voters, is evidently correct but unnecessarily tortuous. The same result – indeed, a more general one – can be reached immediately by focusing directly on the (complete and asymmetric) majority preference relation over platforms.[2]

Consider some platform P_o that is distinct from P★ and thus includes the minority alternative in at least one issue, say B. Now consider the second platform P'_o that includes the majority alternative in B but is otherwise identical to P_o. Obviously P'_o dominates P_o for the attention of all voters is focused on the one issue B with respect to which the two platforms differ and in this respect a majority of voters prefer P'_o. Since the same argument can be made for every platform distinct from P★, the only platform that may possibly be undominated is P★. Thus, if P★ is in fact dominated by at least one other platform (i.e., if a "logrolling situation" exists), every platform is dominated by at least one other platform, which is equivalent to saying that majority preference over platforms is cyclical and that an "Arrow paradox" exists.[3]

This analysis can be readily extended to cover the case of issues with multiple alternatives. Let the k alternatives in issue A be labelled a_1 a_2, etc., in such a way

[2]Majority preference relations can be conveniently represented by directed graphs and I will make use of two elementary graph-theoretic theorems below. See footnotes 3 and 5.

[3]That the non-existence of an undominated platform implies cyclical domination should be directly obvious, but for a formal statement and proof of this point see Frank Harary, Robert Z. Norman, and Dorwin Cartwright, *Structural Models* (New York, 1965), pp. 64-65.

that a majority of voters prefer a_h to a_{h+1} for all h = 1, . . .,k−1.[4] (Obviously, if issue A includes a majority alternative, i.e., is free of a "Type 2 paradox," that alternative is a_1.) And let the alternatives in the remaining issue be labelled in a similar manner.

Every platform other than $(a_1,b_1, . . .)$ is certainly dominated, for any platform including an alternative such as b_h, where $h \neq 1$, is dominated by the otherwise identical platform including b_{h-1}. Also if at least one issue, say B, fails to have a majority alternative because of cyclical majority preference with respect to the alternatives in B (i.e., if there is some b_h that a majority of voters prefer to b_1), then the platform $(a_1,b_1 , . .)$ is certainly dominated (i.e., by the otherwise identical platform including b_h). Finally, even if every issue considered alone is paradox-free, the platform $(a_1,b_1,. . .)$, i.e., P★, may yet be dominated, as Bernholz's discussion of dichotomous (and accordingly paradox-free) issues demonstrates. Thus, the consequence of considering issues demonstrates. Thus, the consequence of considering issues jointly and allowing logrolling agreements across issues may be to create cyclical majorities where none exists within individual issues, *and certainly the incidence of cyclical majorities cannot be reduced by logrolling*. This conclusion clearly and directly contradicts the rather common suggestion that when issues are considered jointly and logrolling is allowed, the possibility of an "Arrow paradox" is precluded or at least reduced.[6]

We see that the existence of a "logrolling situation" implies the existence of an "Arrow paradox." Cyclical majority preference need not entail a "logrolling situation." However, the existence of an "Arrow paradox" of the type that precludes an undominated platform – that is, the existence of what David Klahr has called a "Type 2 paradox"[5] – does imply the existence of a "logrolling situation", for the existence of a "Type 2 paradox" means that every platform is dominated, thus that P★ is dominated, and thus that a "logrolling situation" exists.

It is worth pointing out that the argument in this note is not entirely new. Indeed, a similar argument was made by Joseph B. Kadane in an article that appeared in this journal shortly before Bernholz's article.[7] At about the same time Joe A. Oppenheimer wrote a paper stating the basic result.[8] And several years earlier, R. E. Park formulated the problem of "vote trading" essentially in the manner presented here and reached the same conclusion.[9] More recently, Claude Hillinger demonstrated that the platform P★ may be dominated and noted that an

[4]Such a labelling is always possible because, when individual preferences are strong and N is odd, majority preference generates a "complete asymmetric directed graph" and every such structure has a "complete path"; again see Harary, *et. al., op. cit.,* pp. 295-296.

[5] "A Computer Simulation of the Paradox of Voting," *Ameri-Political Science Review*, Vol. 60, No. 2 (June, 1966), p. 385.

[6]This argument apparently was first made by Gordon Tullock. See James Buchanan and Gordon Tullock, *The Calculus of Consent* (Ann Arbor, 1962), pp. 330, 332. It should be noted that the present formulation of the logrolling problem is rather different from Tullock's formulation.

[7]"On Division of the Question," *Public Choice*, Vol. 13 (Fall, 1972). See in particular Theorem 1 (i) on p. 49.

[8]"Relating Coalitions of Minorities to the Voters' Paradox or Putting the Fly in the Democratic Pie" mimeo, 1972.

[9]"The Possibility of a Social Welfare Function: Comment," *American Economic Review*, Vol. 57, No. 5 (December, 1967), pp. 1300-1304.

example of this possibility (identical to Bernholz's case) "incidentally illustrates the voting [Arrow] paradox"; but (because, in a second example, he did not consider all platforms) he incorrectly suggested that logrolling does not always imply a paradox.[10]

Finally, this problem was formulated in two earlier and better known works, which did not, however, reach the present conclusion. In his discussion of "minorities rule" in *A Preface to Democratic Theory*, Robert Dahl noted (in effect) that the platform P★ may be dominated, though he did not draw the conclusion that majority preference over platforms must then be cyclical.[11] And in his discussion of the "basic logic of government decision-making" in *An Economic Theory of Democracy*, Anthony Downs noted, first, that a "coalition of minorities" may be effective against the platform P★ and, second, that no platform can be undominated if any issue includes an "Arrow problem [paradox]"; however, he did not explicitly note that an effective coalition of minorities implies cyclical majority preference over platforms in every case.[12]

We may note in conclusion that there is some irony in the fact that students of the American political process, on the one hand, have very typically emphasized the importance of logrolling and coalition formation but, on the other hand, have very typically dismissed the "Arrow paradox" as little more than a mathematical curiousity[13] or have ignored it entirely.[14] We see that in fact the two phenomena are logically bound together.

[10] "Voting on Issues and Platforms," *Behavioral Science*, Vol. 16, No. 6 (November, 1971), especially p. 565.

[11] (Chicago, 1956), p. 128. Dahl, however, was aware of the "Arrow Paradox"; see *ibid.*, pp. 42-44n.

[12] (New York, 1957), pp. 55-62.

[13] See, for example, Robert A. Dahl and Charles E. Lindblom, *Politics, Economics, and Welfare* (New York, 1953), pp. 422-423.

[14] One exception to this generalization is provided by John C. Blydenburgh, "The Closed Rule and the Paradox of Voting," *Journal of Politics*, Vol. 33, No. 1 (February, 1971), pp. 57-71.