MINORITIES VS. MAJORITIES:

THE LOCATION OF THE MEDIAN IN A MIXED NORMAL DISTRIBUTION

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July 1998
Revised July 2003

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In doing political analysis in the "Downsian" style, it is often reasonable to suppose that we have a distribution of voter ideal points over an ideological or policy dimension that is composed of two distinct normal distributions, each with its own size, center, and dispersion. For example, the two curves may represent the distribution of ideal points for two groups of partisans (e.g., Democrats and Republicans, Labourites or Conservatives, etc.) — either members of the mass electorate (as in Merrill et al., 1997) or of party caucus in Congress or Parliament (as in Grofman et al., 1999). Most generally, the two curves can represent the distribution of ideological or policy ideal points within any distinctive "minority" group and the complementary "majority" group (as in Miller, 1996). Here I generally refer to the two normally distributed groups as "parties," one with minority and the other with majority status.

In the context of Downsian analysis, the location of the median in each party group and in the overall distribution takes on special significance. This is because Duncan Black's *Median Voter Theorem* asserts that the median location corresponds to the policy or ideological position that a voting group will (tend to) enact or elect if it operates under majority rule. Thus determining the location of the median of any distribution of ideal points is of key importance in the theory of political choice.

It is straightforward to locate the median within each party; given that each party distribution is normal (and thus symmetric), as the median is identical to the mean, i.e., the center of the distribution. It is also straightforward to locate the mean of the overall mixed normal distribution, as it is the average of the two party centers weighted by their sizes (and is independent of the party dispersions). However, locating the median in the overall mixed distribution is much less straightforward, largely because it does depend on the party dispersions as well as the centers and sizes.

Miller (1996) offers some qualitative insights into the location of the median in a mixed normal distribution. Merrill et al. (1999) examine the same problem quantitatively, using a method of approximation that is elegant and quite accurate in many circumstances (i.e., when there is "sufficient overlap" between the two party distributions) but not all, and they demonstrate that a "concentrated minority" can shift the overall median far in its direction. Here I present a fairly simple (but not especially elegant) numerical and graphical procedure that accurately locates the median of a mixed normal distribution under all circumstances, which I use to confirm, refine, and extend propositions about the location of the median of a mixed normal distribution and its dependence on the parameters of the two party distributions.

1. Overview

I first introduce some simple notation. X is one-dimensional ideological or policy space and x is a representative point on X. Party 1 has a *size* p_1, a *center* m_1 (a point on X which is both the median and mean of the ideal points) and a *standard deviation* s_1. The parameters for Party 2 are labelled in parallel manner.

We may reduce these six parameters to three, by establishing a couple of notational conventions. First, we choose a vertical scale so that the area under the two curves (the size of the overall distribution) sums to 1, i.e., $p_1 + p_2 = 1$. Thus we can specify the sizes of both parties by a single parameter P, which we set so that so that $p_2 = P$ and $p_1 = 1 - P$. (Except when we want leave open the question of majority versus minority party status, we will follow the convention that Party 2 is the majority party, i.e., that $P > .5$.) Likewise we choose our horizontal units on the ideological or policy scale so that $m_1 = 0$ and $m_2 = 1$. It then follows that the *mean* m of the *overall distribution* is $p_1 \times m_1 + p_2 \times m_2 = (1 - P) \times 0 + P \times 1 = P$.

Our objective is to specify the location *median* m^* of the *overall distribution*, as a function of the three parameters P, s_1, and s_2. Clearly m^* lies in the interval [0,1] (as does m; if we stipulate that Party 2 has majority status, m lies in the interval [0.5,1]). A subsidiary objective is to specify when and by how much the median lies above or below the mean (and also the midpoint between the two party centers), as a function of the same parameters.

The political choice problem is to chose a point in the [0,1], and we suppose that this is the median point. Three other points in the [0,1] interval are especially significant in political terms. We consider these in order from highest to lowest.

The first salient point is 1, i.e., the *center of the majority party*. Lani Guinier (among many others) has discussed archetypical examples of political choice in which majority rule implies that “the numerically more powerful majority choice simply subsumes minority preferences” (1994, p. 2) — in effect that collective choice may be the majority center. Indeed, we can have $m^* = 1$, but only under one quite special circumstance — namely, that the majority party is totally concentrated (whether resulting from ideological cohesion or total discipline within its caucus), i.e., has zero dispersion (in which case neither the size nor the cohesion of the minority party has any impact on the location of m^*). But in all other circumstances, m^* is located at least slightly below 1. Miller in particular focuses on the extent to which, and conditions under which, the minority has "impact" on collective choice, i.e., shifts m^* away from the majority center.

The second salient point in the [0,1] interval is m (= P), i.e., the *mean of the overall distribution*. The mean may suggest itself as a "fair compromise" between 0 (the center of the minority party) and 1 (the center of the majority party). It is indeed a compromise, in that it is always in the interior of the [0,1] interval, and it reflects the respective numbers of voters in each party. Moreover, if we have an election in which all voters vote for either 0 or 1 according to their party affiliation, and if political choice is made not by majority rule but by random selection of one ballot (arguably more equitable than majority rule based on all ballots; see Wolff, 1970, pp. 45-47), the *expected* political choice is m. Thus it is of interest to determine the conditions under which m^* lies above, at, or below m.

The third salient point in the [0,1] interval is 0.5, i.e., the *midpoint between the two party centers*. This is also a true compromise, but one that neglects to take into account the respective numbers of the two parties, in which sense it is relatively favorable to the minority. The point 0.5 has particular empirical significance if we suppose (in the manner suggested by Merrill et al.) that voters in each party select party nominees with positions corresponding to their party centers, i.e., 0 and 1 respectively, in something like U.S.-style primary elections, and that these nominees are subsequently paired in a general election. If, in the general election, voters vote for the candidate closest to them ideologically (irrespective of their party affiliation), the candidate of the majority party (with position 1) wins if $m^* > .5$ and the candidate of the minority party (with position 0) wins if $m^* < .5$.

2. Procedure

Consider any point x on the ideological scale such that $0 \leq x \leq 1$. For any such x (and provided that both parties have non-zero dispersion) , we can partition the area under the Party 1 curve into that portion A_1 that lies below 0, that portion B_1 that lies between 0 and x, and that portion C_1 that lies above x. Likewise, we can partition the area under the Party 2 curve into that portion A_2 that lies above 1, that portion B_2 that lies between x and 1, and that portion C_2 that lies below x. (See Figure 1. Note that C_1 may include area above 1 and C_2 may include area below 0.)

Using this notation, we can now do some very simple algebra. For any x, we have:

$$A_1 + B_1 + C_1 = p_1 = 1 - P$$

and also

$$A_1 = 0.5 \times p_1 = 0.5 \times (1 - P)$$

so

$$C_1 = P_1 - A_1 - B_1 = (1 - P) - 0.5 \times (1 - P) - B_1 = 0.5 - .5 \times P - B_1$$

Likewise, we have:

$$A_2 + B_2 + C_2 = P$$

and also

$$A_2 = 0.5 \times P$$

so

$$C_2 = P - A_2 - B_2 = P - 0.5 \times P - B_2 = 0.5 \times P - B_2$$

Point x coincides with the median m^* of the overall distribution if it partitions areas under the two curves so that:

$$A_1 + B_1 + C_2 = 0.5 \ (= C_1 + B_2 + A_2)$$

Substituting for A_1 and C_2, the x is the median m^* if:

$$0.5 - 0.5 \times P + B_1 + 0.5 \times P - B_2 = 0.5$$

$$B_1 = B_2 \qquad (1)$$

Thus *we can locate the median m* of the overall distribution by finding the value of x that produces B areas of equal size*.

Let $\Phi[z]$ be the relative area under the normal curve from its mean to z, where z is expressed in standard units. Thus x is the median m^* if and only if:

$$\Phi[m/s_1] \times (1 - P) = \Phi[(1-x)/s_2] \times P \qquad (2)$$

(This is essentially Lemma 1 in Merrill et al., 1999.) Given any P, s_1, and s_2, we can in principle find the value of x that produces the equality given above.

In Figure 2, we plot the relationship between $\Phi(z_1)$ and z_1 (the upward sloping curve) as well as the relationship between $\Phi(z_2)$ and z_2, where $z_2 = 4 - z_1$ (the downward sloping curve).

In principle, we can determine m^* for any P, s_1, and s_2 directly from Figure 2. In general, however, we must do two things first. First, the $\Phi(z)$ curves must be rescaled vertically to reflect the relative magnitudes of $1 - P$ and P. Second, the $\Phi(z)$ curves must be both shifted and truncated horizontally to reflect the magnitudes of s_1 and s_2. In practice, it is more convenient to transform the variables in an appropriate manner and then produce a new graph.

For example, suppose that $P = .6$ (the majority party includes 60% of the electorate) and that $s_1 = 0.5$ and $s_2 = 1$ (the minority party is more cohesive but there is substantial overlap between the parties). First, since $P = .6$, we must vertically rescale the curves so that $\Phi(z_1)$ approaches a maximum of 0.3 and $\Phi(z_2)$ approaches a maximum of 0.2 (rather than both approaching a maximum of 0.5). Second, since $z_1 = x/s_1 = x/.5 = 2x$, we are concerned only with that portion of the $\Phi(z_1)$ curve that runs from $z_1 = 0$ to $z_1 = 2$. Likewise, since $z_2 = m/s_2 = m/1 = x$, we are concerned only with that portion of the $\Phi(z_2)$ curve that runs from $z_2 = 0$ to $z_2 = 1$. Superimposing (the relevant portions of) the two curves over the X space running from $x = 0$ to $x = 1$ produces Figure 3A. It is apparent the two curves intersect (implying that $B_1 = B_2$) at approximately $x = 0.45$. We can "zoom in" on the relevant portion of the graph, as shown in Figure 3B, to determine quite precisely that $x \simeq .4475$ (the value being accurate at least to the third decimal place).[1]

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[1] All results and graphs presented here were produced by SPSS for Windows. The original "data" is the single variable Z_1, running from 0 to 4 in increments of .01 (and extending on to 7 in larger increments). Other variables are transformations derived from Z_1 that incorporate the parameters P, s_1, and s_2. These other variables are:

```
NORM1 = CDF.NORMAL(Z1,0,1) - 0.5
X = Z1 * S1
Z2 = 1 / S2 - Z1 * S1 / S2
NORM2 = CDF.NORMAL(Z2,0,1) - 0.5
WGTNORM1=NORM1*(1-P)
WGTNORM2=NORM2*P
```

We may note that this example illustrates the phenomenon alluded to earlier that Merrill et al. call “the power of ideologically concentrated minorities.” The greater concentration of the minority Party 1 advantages it to the extent that of the median position m^* not only lies below then mean of 0.6 but is actually closer to the minority center than the majority center. Two obvious follow-up questions arise. First, fixing these values of s_1 and s_2, how much smaller could Party 1 be and m^* remain below 0.5 (or the mean of 0.6)? Alternatively, fixing Party 1's size at 0.4, how much more dispersed could it become — or how much more concentrated could Party 2 become — and m^* remain below 0.5 (or the mean of 0.6)?

By constructing similar graphs at varying sizes of P, s_1, and s_2, we can trace out the effects of variation in party size and dispersion on the location of the median and answer such follow-up questions.

3. Results

The exact location of m^* is evident in at least two special cases. First, as previously noted, if the majority party is totally cohesive ($s_2 = 0$), $m^* = 1$. Second, if the two parties are exactly the same size ($P = .5$) and have exactly the same dispersion ($s_1 = s_2$), regardless of the level of that common dispersion, the overall distribution is symmetric, so $m^* = m = 0.5$. Moreover, it is apparent that m^* “typically” lies between m and 1; indeed, this is always true provided the minority party is no more concentrated than the majority party, i.e., $s_1 \geq s_2$). However, if the minority party is sufficiently more concentrated than the majority, we can get $m^* < m$ and even $m^* < .5$ (as illustrated in by the example in Figure 3). Merrill et al. reached important insights about how party size and dispersion trade-off against each other in determining the location of m^*. Using the procedure described above, we can be both more precise and more general.

First let us consider the situation in which the parties have equal dispersion at varying levels. Figure 4A displays the location of the overall median m^* (on the vertical axis) as a function of P (PSIZE on the horizontal axis) for five different levels of party dispersion, $s = 0$, $s = 0.15$, $s = 0.3$, $s = 0.5$, and $s = 1$ (identified in the legend as S000, S15, S30, S50, and S100). The marks on each plotted curve in Figure 4A (and subsequently) represent actual values of m^*, as determined by the

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Each graph such as Figure 3 is an overlay scatterplot of WGTNORM1 by X and WGTNORM2 by X, from which point of intersection between the two curves can be read from the horizontal scale. Note that we can also read from the vertical scale the B value that equates B_1 and B_2, in this case B = .1258. Simple addition allows us to verify that this value is correct:

A_1 =	0.5×0.4 =	0.2000		0.2000	
B_1 =			0.1258	0.1258	
C_1 =	0.2 − B =	0.0742			0.0742
A_2 =	0.5×0.6 =	0.3000		0.3000	
B_2 =		0.1258		0.1258	
C_2 =	0.3 − B =	0.1742	0.1742	______	
		1.0000		0.5000	0.5000

procedure described above. These marks are connected by simple linear interpolation. The heavy diagonal straight line (0,0) to (1,1) plots the location of the mean, i.e., the line $x = P$.

If there is no variance within the parties at all (producing what Miller calls "cleavage politics"), the median is simply the center of the larger party, i.e., either 0 or 1. Thus the median curve for $s = 0$ (labelled S000) runs level from (0,0) to (0,0.5), jumps discontinuously to (0.5,1) and then runs level to (1,1).

But if there is any dispersion within the parties at all, the plot of the location of the median m^* as a function of P becomes a smooth S-curve that begins at (0,0), passes through (.5,.5), ends at (1,1), and rises monotonically throughout. Below $P = 0.5$, all median curves fall below the diagonal mean line; above $P = .5$, all median curves rise above the mean line. The magnitude of the deviation of the median curve from the mean line depends on the degree of cohesion or concentration within the parties.

Even small dispersion within the parties produces a distinct S-curve. The curve for $s = 0.15$ (which implies essentially zero overlap between the parties, as the distance between the two party centers approaches 7 standard deviations) is displayed in Figure 5, along with those for s = .3 (still below Miller's "critical threshold" of about s = .4), s = .5 (the threshold of Merrill et al.'s condition of "sufficient overlap"), and s = 1. As dispersion within the parties increase, the median S-curve flattens out and, at the limit, converges on the mean line. Moreover this limit is approached quite rapidly; even at s = 1 (which still implies substantial differentiation between the parties), the median curve is hardly distinguishable from the mean line.

The northeast quadrant of Figure 4A is identical to the southwest quadrant pivoted 180° about the point (.5,.5) and the other two quadrants are empty. Thus, in this symmetric situation, we can focus exclusively on the northwest quadrant (and conform with our usual convention that P > .5). Figure 4B blows up this quadrant for easier reading.

Figures 5A and 5B present the same information from a different perspective by plotting the deviation of m* from $\overline{m}$ as a function of P. As party dispersion diminishes, the (positive or negative) deviation increases and the P value that maximizes the deviation drifts towards .5 (basically because the potential deviation from the mean in the admissible direction -- downward for P < .5 and upward for P > .5 — increases as P approaches .5).

These figures can interpreted in a another way. In many political contexts, it is meaningful to talk about parties being convergent or polarized -- that is, to suppose that the absolute distance between party centers (somehow reckoned) may vary. (Miller explicitly allows for this.) Here (following Merrill et al.), we have standardized this distance at 1 unit. However, we can also read Figure 4A and similar charts as indicating how the location of the median shifts as (absolute) distance between the party centers changes while (absolute) party dispersion remains constant -- and therefore dispersion <u>relative to the distance between the party centers</u> changes). Nearly complete convergence between the parties is equivalent to very large dispersion, and results in $m^* \approx \overline{m} = P$. Modest differentiation between the parties is equivalent quite large dispersion, so the location of m* relative to the distance between party centers is suggested by the S = 1 curve. At the other extreme, great polarization between the parties is equivalent to small dispersion, so the location of m* relative to the distance between party centers given by a curve such as S15.

In the latter connection, it is worth taking explicit note of the fact that, when dispersion is small (and/or polarization is great), the resulting median curve is close to vertical as it passes through the central point (.5,.5). This theoretical point underlies the empirical fact emphasized by Grofman et al. that, as the Democratic and Republican House caucuses have in recent years become both more liberal and conservative respectively (the parties have polarized) and also less dispersed (as moderate-to-conservative Democrats and moderate-to-liberal Republicans have disappeared), <u>very small shifts in the relative sizes of the parties in the vicinity of P ≈ .5 produce enormous changes in the location of the House median</u>.

We now allow dispersion to differ between the parties. Figure 6 is parallel to Figure 4A, except that dispersion in Party 1 is held constant at $s_1 = .3$ (quite low dispersion), and we plot the location of m* as a function of P for different levels of dispersion in Party 2. One plotted level of dispersion in Party 2 is $s_2 = .3$, so in this case the parties have equal dispersion, and the dashed S30S30 line in Figure 6 is identical to the dashed S30 line in Figure 4A.

We see that if Party 2 is less dispersed than Party 1 ($s_2 = .15$), the median line starts out from (0,0) in essentially the same manner as in the symmetric ($s_2 = .3$) case, but as P approaches .5 (as Party 2 approaches majority size), the median moves more sharply upward. The curve just intersects the northwest quadrant of the graph, so that m* > .5 even as P < .5. Above about P = .52, the S30S15 curve in Figure 6 closely duplicates the S15 curve in Figure 4. On the other hand, when dispersion of Party 2 increases above that of Party 1, the location of m* drifts downward. If $s_2 = .5$ (the S30S50 line), m* falls substantially below .5 when P = .5 (m* ≃ .375); m* doesn't reach .5 until about P ≃ .625; and m* doesn't exceed $\overline{m}$ until about P ≃ .66, and it never exceeds $\overline{m}$ by much. If $s_2 = .7$ (the S30S70 line), the median curve is dragged further down; m* doesn't reach .5 until about P ≃ .69; and m* barely exceeds $\overline{m}$ only when P is well over .9 . And the S30S100 ($s_2 = 1$) and S30S130 ($s_2 = 1.3$) median curves never cross the mean line.

Figure 7 corresponds to Figure 5A by focusing on the deviation of m* from $\overline{m}$. Again the DIFS30 line in Figure 8 is identical to the DIFS30 line in Figure 6A.

We can generalize Figure 6 by considering many different combinations of party dispersion. The ideal arrangement would be to display the level of m* (for different levels of P) as a surface of varying height over a square s_1 by s_2 plane. In the absence of such sophisticated graphics, Table 1 (supplemented by Figures 8, 9, and 10) conveys similar information for the case of P = .6 (i.e., a moderate imbalance in size between the parties).

Table 1 is a matrix the rows of which correspond to different levels of dispersion in the minority party (SMIN), the columns of which correspond to different levels of dispersion in the majority party (SMAJ), and the cells of which display the resulting value of m* (truncated to two decimal places) when P = .6 .

Figure 8 plots the location of m* over in each row of the matrix, i.e., it displays horizontal cross-sections of the m* surface in a three-dimensional graph. More substantively, each curve displays the location of m* over varying levels of majority party dispersion at a specified level of minority party dispersion. Necessarily the figure omits the final m* = .00 entry in each row, achieved when the majority party has indefinitely large dispersion, since such a value of s_1 cannot be placed on an interval scale.

We see from Figure 8 what has been alluded to several times already -- that is, if the majority party has no dispersion, m* = 1 regardless of dispersion in the minority party. Invariably, the location of the median drifts in the direction of the minority center as dispersion in the majority party increases, but this drift is slow if minority dispersion is large and abrupt if minority dispersion is small. The limiting case is a totally concentrated minority (as indicated by the S000) line, which deserves special mention. It can be seen that m* falls linearly from 1 to 0, as s_2 (SMAJ) increases from 0 to about 1.034 but then can fall no further. At $s_1 = s_2 = 0$, 40% of the electorate is concentrated at point 0 and 60% at point 1. When s_2 is positive (but not too large), m* is the number m that makes the area under the majority curve to the left of m equal to .1 of the total area, which when added to the .4 minority constitutes .5 of the overall distribution. This .1 of the total electorate constitutes 1/6 of the majority and 1/3 of the majority below the majority center. Thus m* is always .967 standard deviations away the majority center, and accordingly is a linear function of s_2. However, once s_2 exceeds 1.034 (= 1/.967), m* = 0 and it is held there by the 40% of the electorate concentrated at the point, even as s_2 increases further.

Figure 9 plots the location of m* over in each column of the matrix, i.e., it displays vertical cross-sections of the m* surface in a three-dimensional graph. More substantively, each curve displays the location of m* over varying levels of minority party dispersion at a specified level of majority party dispersion. Again necessarily the figure omits the final m* = 1.00 in each column, achieved when the minority party has indefinitely large dispersion.

In both Figures 8 and 9, I have drawn in the same additional curve that crosses other curves and asymptotically approaches the horizontal line m* = .6. This plots the intersection of each curve corresponding to a given level of dispersion for one party with the vertical line corresponding to the same level of dispersion for the other party. Put another way, it plots the m* values that run down the main diagonal of Table 1. And put substantively, it shows what happens to m* when both parties have the same dispersion and that common dispersion increases (and/or polarization decreases); m* rapidly approaches $\overline{m} = P$. This curve thus conveys information similar to Figures 4A and 4B, though here we take one particular value of P and let s vary continuously, where Figure 4 takes selected values of s and let P vary continuously.

Finally, Figure 10 arranges the data in Table 1 (again excluding the indefinitely distant row and column) in a manner that reflects the interval properties of SMIN and SMAJ. I have then sketched in approximate contour lines for values of m* at intervals of .1 unit, from which a clear pattern of m*-elevation emerges.

The next figures trace, over all combinations of party dispersion, the size (P) of the majority party such that m* = .5, i.e., the P-threshold that separates majority party victory from defeat, as in the Merrill et al. setup.[2] Figure 11 displays the approximation proposed by Merrill et al., according to which the threshold is given by the ratio $s_1/s_2 = p_1/p_2$, so $s_1 = s_2 \times p_1/p_2$. For example, the heavy solid line marked R65 tells us that (according to the approximation), if the majority party with 65% of the electorate has dispersion $s_2 = 1.5$, it can win provided minority party dispersion s_1 exceeds about $1.5 \times .35/.65 \simeq .81$. As Merrill et al. are careful to note, this approximation is good only if there is "sufficient overlap" between the two parties, which they fix at about $s_1 + s_2 \geq 1$. This constraint is shown in Figure 1, so the shaded portion of the graph is not applicable.

Figure 12 is identical to Figure 11 except that it plots exact values of m*, accurate for all values of party dispersion. Clearly Figure 11 is very similar to Figure 10 except near the horizontal axis. It appears that the condition for the approximation to be good depends less (or not at all) on the sum of s_1 and $s_2 \geq 1$ and more (or entirely) on the magnitude of s_2 alone, with $s_2 \geq .35$ or so being the threshold. It is worth thinking out what is happening as each curve "turns vertical" -- for example, at the point $s_1 \approx .12$ and $s_2 \approx .68$ on the line for P = .65 (S1WINP65). When $s_2 = .68$, m = .5 is .5/.68 = .74 SDs away from the majority center. From a normal table, it can be checked that about .27 of the electorate of the majority party lies within this range and thus about .77 lies above .5, which is $.77 \times .65 \simeq .50$ of the total electorate. Thus at this level of dispersion a 65% majority can just win on its own, regardless of the dispersion in the minority party. Moreover, at $s_1 \simeq .12$, .5 is 4 standard deviation above the minority party center, so the minority contributes (essentially) no votes to the majority side if it is concentrated to this degree or more.

The location of the points at which each curve intersects the horizontal axis tells us that a majority party of the specified size with at least that degree of concentration can win in the Merrill et al. setup, regardless of the concentration of the minority party. That is, a majority of P = .52 is guaranteed victory with $s_2 \leq .38$, of P = .55 with $s_2 \leq .47$, and so forth.

Figure 13 is directly comparable to Figure 12, except that it traces, over all combinations of party dispersion, the size of the majority party such that $m^* = \overline{m}$, i.e., the P-threshold that determines whether the median falls above or below the mean.

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[2] The variables are defined as follows:

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ZWIN = .5 / S2
ZMEAN = P / S2
NORMZW = CDF.NORMAL(ZWIN,0,1) - .5
NORMZM = CDF.NORMAL(ZMEAN,0,1) - .5
WGTNZW = NORMZW * P / (1-P)
WGTNZM = NORMZM * P / (1-P)
WGTNZW5 = WGTNZW + .5
MINWIN = PROBIT(WGTNZW5)
WGTNZM5 = WGTNZM + .5
MINMEAN = PROBIT(WGTNZM5)
S1WIN = .5 / MINWIN
S1MEAN = (1-P) / MINMEAN
```

4. Summary

As we have seen, determining the location of the median in a mixed normal distribution is far from straightforward. Here I attempt a qualitative but precise summary.

The location of the median in a mixed normal distribution in the interval [0,1] is a functions of three parameters -- P, s_1, and s_2. It may be more intuitive to redefine the dispersion parameters and say that the location of the median is a function of (i) the disparity of size between the majority and minority parties, (ii) the average dispersion within the two parties, and (iii) the disparity in dispersion between the majority and minority parties.

Other things being equal, the following statements are almost always true (and their opposites are never true). The median shifts in the direction of the center of the majority party center, (i) the bigger the size advantage of the majority party, (ii) the smaller the average dispersion within the two parties, and (ii) the smaller the dispersion of the majority party relative to the minority party. Under the opposite conditions, the median almost always shifts in the direction of the center of minority party (and never shifts in the other direction).

These effects are in no way linear additive, however; the effects of the three parameters interact in a variety of ways. First, the median is constrained to the interval between the two party centers and, once we get "boundary" solution, the median can shift no further in the expected direction. For example, the median is identical to the center of a totally concentrated majority party and thus cannot shift further toward the majority party, even if the size majority party increases or its dispersion decreases. (Conversely, recall the plot of S000 in Figure 8).

In addition, there are "near boundary" solution, in which further changes in a parameter have <u>almost</u> no further effect. The substantive message in Merrill et al.'s approximation method (related also to Miller's "critical threshold" notion) is that the second parameter (average dispersion within the two parties) makes almost no difference once there is "sufficient overlap" -- a point confirmed by the fact that the curves in Figure 12 quickly approximately a linear form (though we can improve on the condition for this linearity).

The location of the <u>mean</u> in a mixed normal distribution <u>is</u> a simple linear function of the first parameter only, i.e., $\overline{m} = P$. The median converges on the same simple linear function if party dispersions are sufficiently large (and/or convergence sufficiently complete); however, the median is very close to the mean whenever party dispersion are (more or less) equal and at least moderately (e.g., s = 1) large. As the parties become (more or less equally) more concentrated (and/or more polarized), the median moves (relatively) closer to the center of the majority party, and it moves in that direction more rapidly the more closely balanced the parties are in size.

If the disparity in dispersion between the parties changes, the median shifts in the direction of the more concentrated party. But the baseline from which this shift begins is one that is favorable to the majority party. If the parties have equal dispersion, the median is always between the mean (a simple function of the majority's size advantage) and the center of the majority party -- closer to the mean if they have large equal dispersion and closer to the majority center if they have smaller equal dispersion. If the majority party becomes more concentrated, the median moves still closer to its center. But if the minority party becomes more concentrated, the median moves toward the minority center. If the disparity in dispersion becomes sufficiently favorable to the minority (how much depending on the size of the minority and on average dispersion, especially if this average is fairly low), the median can move into the interval between the mean and the center of the minority party. If the disparity in dispersion becomes sufficiently more favorable to the minority (how much again depending on the size of the minority and now on the dispersion of the majority party, especially if this average is fairly low), the median can move into the interval between .5 and the center of the minority party. If the minority party is totally concentrated and/or its size approaches .5 and majority dispersion becomes sufficiently large, the median can equal the minority center.

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