COMPLEXITIES OF MAJORITY VOTING

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1. Overview

If a group of people needs to make a collective decision of some sort, it is common for someone to say something like 'Let's vote on it — and the majority rules'. However, the group may run into difficulties, because it turns out that majority voting presents unexpected complexities. This module in the *Vote*Democracy course examines these complexities.

The *Vote*Democracy course draws on scientific work on the theory of voting and social choice to examine the components of democratic theory and practice that entail voting. With respect to the more common 'indirect' or 'representative' variant of democracy, this entails both *voting for representatives* (to parliament or other assemblies) or other officials and *voting by representatives* (within parliament or other assemblies). With respect to the less common 'direct' variant of democracy, this entails voting in referendums. Most such voting is broadly 'majoritarian' in character though, as we shall see, this term has various meanings.

Sections 1 through 6 of this module examine general concepts and problems pertaining to majority voting, including the 'majority preference relation' and its potentially 'cyclical' nature. Section 7 describes a number of specific voting rules, all of which are broadly majoritarian in nature, for selecting one alternative out of several. In this context, the alternatives are typically *candidates* — that is to say, the voting rules pertain to voting for representatives elected from single-member districts or for executive officials. Section 8 assesses these rules in terms of various properties that may be deemed desirable. The remaining sections examine voting rules that entail a sequence of pairwise majority votes; such rules include voting of the parliamentary type — that is to say, voting by representatives, where the alternatives are not candidates but 'motions', 'bills', 'amendments', etc. However, parliamentary voting actually entails sequence of 'yes' or 'no' votes and takes place under a particular *agenda*, i.e., a specified order in which the votes take place. Such agendas are broadly of two types: 'amendment agendas' of the sort generally used in Anglo-American legislative bodies, and 'successive agendas' more commonly used in continental European parliaments.

2. Majorities and Minorities

It is common in many kinds of political discourse to refer to 'the majority' and 'the minority' as if each were a unique and identifiable group. This usage may be justified if 'majority' and 'minority' are defined in sociological (e.g., racial, ethnic, religious, class, etc.) terms. This usage in turn may suggest that 'majority rule' is problematic as a political institution, as it appears to divide a polity into permanent winners ('the majority') and losers ('the minority') in a way that seems to undermine political legitimacy and stability.

However, here we define 'majority' and 'minority' in terms of *preferences* that citizens hold, or the way they vote, on *political issues* (or candidates or parties). On this interpretation, 'majority rule' can be problematic in that unexpected complexities — which may rise to the level of 'paradoxes' — arise but, at the same time, these complexities may have the effect of making majority rule less problematic as a political institution. These complexities arise because, in all but the simplest circumstance, it is likely that several or many distinct majorities (and minorities) exist with respect to preferences on one or more issues.

Specifically, a unique 'majority' and 'minority' must exist only in the case of preferences on a *single issue* that presents exactly *two options* in the manner of a typical *referendum* (see Box X), one group being defined as those who prefer, or vote for, the first option (e.g., 'yes') and the other as those who prefer, or vote for, the second option (e.g., 'no'). Unless two groups are precisely the same size (and setting aside those who have no preference or cannot or do not vote), one constitutes 'the majority' and the other 'the minority' with respect to this single two-sided issue.

However, if there are two or more issues, or if a single issue presents more than two options, it is likely that multiple majorities exist with respect to the different issues or different pairs of options on the single issue. Given a fixed set of voters, it is of course an arithmetical necessity that any two majorities overlap, i.e., are 'intersecting sets' (see Box X on Set Theory), but it is quite likely that no overall majority with common preferences over the several issues or options exists.

In order to formalize these arguments, we introduce some basic terminology and assumptions drawn from the theory of voting and social choice. There are two basic building blocks: a set of *alternatives* that correspond to the several options on a given issue, to combinations of options on multiple issues, or to candidates in an election; and a set of *voters* each of whom has *preferences* over the alternatives.

Each voter's preferences are described as a (*strong* or *weak) ordering* that ranks the alternatives from most preferred to least preferred. A *strong* ordering rules out indifference between any pair of alternatives, i.e., ranking alternatives at the same level of preference, while a *weak* ordering allows indifference. This assumption implies that each voter's preferences are *complete*, i.e., for every pair of alternatives x and y, each voter prefers x to y, prefers y to x, or is indifferent between x and y, and *transitive*, i.e., a voter who regards x as at least as good as y and y to z also regards x as at least as good as z. (See Box X [on Binary Relations].) In this introductory module, we generally assume that voters have strong preference orderings.

A set of preference orderings, one for each voter, is called a *preference profile* and describes 'public opinion' for a group of voters with respect to a set of alternatives. An *anonymous* profile merely specifies the *popularity* of each ordering, i.e., how many voters have each ordering, without specifying the orderings of individual voters.

We suppose that a *social choice* must be made from among these alternatives using some *voting rule*, and we focus primarily on variants of majority rule. Under any voting rule, voters declare something about their preferences, perhaps by submitting *ballots* that rank all alternatives in a (strong) declared order of preference. While many voting rules do not make use of full ballot rankings (and some proceed through multiple rounds of voting), it is convenient to suppose that rankings are submitted in any case. However, some voting rules require preference information that cannot be inferred from ballot rankings. A set of ballots, one for each voter, is called a *ballot profile*. Unless noted otherwise, we assume that voters vote *sincerely* — that is to say, that each voter's ballot ranking is identical to the voter's (strong) preference ordering.

Given a ballot profile, a voting rule determines the social choice or *voting outcome*, i.e., the chosen or winning alternative. However, voting rules — especially variants of majority rule — can produce 'tie outcomes' of various sorts, which must then be broken by some further rule. In this introductory module, we will mostly sidestep this problem. The question arises whether or to what extent different voting rules — and, in particular, different variants of majority rule — may produce different voting outcomes from the same ballot profile and what this range of possible outcomes may be. Another question is whether and when it may be that voters have incentives to submit 'insincere' or 'strategic' ballot rankings that differ from their true preference orderings.

Bibliographical Notes and Further Readings

General discussions of majorities, minorities, and majority rule may be found in Barry (1979), Spitz (1984), Chapman and Wertheimer (1990), Miller (1996), and Novak and Elster (2014), among many other works. Many parts of this module draw on the pioneering work of Black (1948, 1958) and Farquharson (1969) and on my own survey (Miller, 1995) of the literature that Black and Farquharson spawned.

3. Majority Voting with Two-Sided Issues

Apart from the problem of ties, majority voting works in a straightforward way given a choice between just two alternatives — for example, a single proposal that can be only accepted or rejected or an election with just two candidates. But problems arise when three or more alternatives are under consideration — for example, when two or more proposals are under simultaneous consideration, when a proposal can be amended as well as rejected, or when an election has three or more candidates. We take up the two-alternative case first and then move to the case of multiple alternatives.

3.1 A Single Two-Sided Issue: May's Theorem

Suppose we have a single issue X with two options x and y. Under most circumstances, simple majority voting strikes most people as the fair and reasonable way to collectively choose between x and y. Many years ago, Kenneth May (1952) formalized this intuition by identifying several conditions that we may want a voting rule to meet, and he then demonstrated that a particular variant of majority rule — and only it — meets these conditions in the two-alternative case. The conditions may be formulated as follows (May formulated them slightly differently):

> *Anonymity* (of voters): the voting rule does not take account of which voter has submitted which ballot ranking in determining the voting outcome, i.e., the rule operates on anonymous profiles.
>
> *Neutrality* (between alternatives): the voting rule does not take account of which alternative represent which substantive proposals (or candidates) in determining the voting outcome.

A condition that is appealing on practical grounds is the following:

> *Resoluteness*. Regardless of how votes are cast, deadlock is avoided and there is always a clear winning alternative.

However, resoluteness is inconsistent with anonymity and neutrality. In the event that an even number of (non-abstaining) voters have equally divided preferences between the two alternatives, anonymity and neutrality together require the kind of symmetric deadlock (i.e., a tie) that resoluteness rules out. Thus we must weaken resoluteness to say that deadlock occurs only as a 'knife-edge' condition.

> *Almost Resoluteness*. Any deadlock is removed if any voter changes his or her vote in any fashion (i.e., from one alternative to the other, or from abstention/indifference to either alternative or vice versa).

In practice, Simple Majority Rule may be rendered resolute by building in some tie-breaking mechanism. Two such mechanisms are common: one breaks a tie in favor of one alternative (typically the one that represents the status quo), thus violating neutrality; another gives one voter (the 'chairmen') a 'casting vote' instead of, or in addition to, a regular vote that is cast only to break a tie, thus violating anonymity.[1]

Finally, votes never count negatively.

> *Non-Negative Responsiveness*. Given a profile under which one alternative (say x) is the winner, if any voter then switches his vote in favor of x (i.e., from y to x, or from abstention/indifference to x, or from y to abstention/indifference), x remains the winner; and, given any profile that produces a deadlock, if any voter switches his vote in favor of x, either the deadlock remains or x becomes the winner.[2]

The variant of majority voting that uniquely meets May's four conditions is *Simple Majority Rule* (SMR). Let $n(xy)$ be the number of ballots that rank alternative x over y. Under SMR voters submit ballots ranking x over y, or y over x, or expressing indifference between x and y; and x is the voting outcome if $n(xy) > n(yx)$, y is the voting outcome if $n(yx) > n(xy)$, and x and y tie as the voting outcome if $n(xy) = n(yx)$.

> ***May's Theorem***. Given two alternatives, Simple Majority Rule meets the conditions of Anonymity, Neutrality, Almost Resoluteness, and Non-Negative Responsiveness and is the only voting rule that does so.

Another variant of majority rule that fails to meet all of May's conditions in the two-alternative case is *Absolute Majority Rule*, under which x is the voting outcome if $n(xy) > n/2$, y is the voting outcome if $n(xy) > n/2$; otherwise, either (i) x and y are deemed to be tied (the usual social choice theory interpretation) or (ii) the alternative that represents the status quo is preserved as the voting outcome. Version (i) violates Almost Resoluteness (and therefore a deadlock cannot always be broken by a single casting vote), while version (ii) violates Neutrality (but is Resolute). Simple and Absolute Majority Rule are equivalent in the event that no voters declare indifference between x and y (or abstain).

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[1] Another possibility is to use a random device (e.g., a coin flip) to break the tie; this preserves generalized versions of the Anonymity and Neutrality condition. However, here we consider only *deterministic* voting rules that do not incorporate a random element.

[2] May combined Almost Resoluteness and Non-Negative Responsiveness into a single condition called Positive Responsiveness.

Majority rule may be generalized to include all variants of *Supermajority Rule* (see Box X), which in practice requires the support of some 'supermajority' of all voters to change the status quo. For example, under two-thirds supermajority rule, x (a proposal to change the status quo) is the voting outcome if $n(xy) > 2n/3$ and y (the status quo) is the voting outcome otherwise. Such a rule obviously violates neutrality by favoring the status quo alternative. In general, a supermajority rule is specified by some *quota* q, where $n/2 < q \leq 1$ and x is the voting outcome if $n(xy) \geq q \times n$, and the limiting cases are Absolute Majority Rule and Unanimity Rule ($q = 1$).[3]

Simple Majority Rule may also be generalized to include variants of Weighted Majority Rule, as used in stockholders meetings (where votes are weighted by ownership shares), many international organizations (where votes are weighted by financial contributions), etc., violates Anonymity by deliberate design.[4]

3.2 Two Two-Sided Issues

Let us now suppose that we have two issues X_1 and X_2, each with two options: x_1 and $\bar{x}_1$ and x_2 and $\bar{x}_2$. Let us further suppose that all voters have strong preferences on both issues, that no ties exist, and that the majority-supported options are x_1 and x_2. Preferences on the two issues partition the voters into four 'clusters'. The *majority cluster* is composed of voters who prefer the majority-supported alternatives on both issues, i.e., x_1 and x_2. The *minority cluster* is composed of voters who prefer the minority-supported alternatives on both issues, i.e., $\bar{x}_1$ and $\bar{x}_2$. Two *mixed clusters* are composed of voters who prefer the majority alternative on one issue and the minority alternative on the other, i.e., either x_1 and $\bar{x}_2$ or $\bar{x}_1$ and x_2.

The relative sizes of these clusters depend on the *popularity* of the two majority-supported options and the degree to which preferences on the two issues are *correlated*, i.e., the degree to which voters who prefer the majority option on one issue also prefer the majority option on the other issue. At one extreme, the majorities supporting x_1 and x_2 may wholly overlap, so that the smaller majority is a subset of the larger one (in sociological terms, preferences on the two issues are 'reinforcing'). In this event, the majority cluster is itself of majority size and at least one mixed cluster is empty. At the other extreme, there is minimal overlap between the majorities supporting x_1 and x_2 (though by arithmetical necessity at least one voter is common to any two majorities) and all other voters belong to mixed clusters. If there is no correlation between preferences on the two issues (in sociological terms, if preferences on the two issues are perfectly 'crosscutting') and if both majorities are of minimum size, the proportion of voters in the majority cluster may barely exceed 25% but in any case it is larger than any other cluster. While the majority cluster is larger than the minority cluster whatever the levels of popularity or degree of correlation, it certainly need not itself be of majority size, so the set of voters who prefer the majority position on both issues may not constitute a majority of voters.

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[3] In social choice theory (see Module 5), such rules are commonly known as *q-rules*, but they are commonly taken as Neutral but (highly) irresolute, in the manner of version (ii) of Absolute Majority Rule

[4] Other examples of weighted voting include the EU Council of Ministers and the US Electoral College. Weighted voting is a principal focus of Module 15.

3.3 Multiple Two-sided Issues

Now let us suppose that we have three or more issues *X*, *Y*, *Z*, etc., each with two options, and that we define majority and minority clusters as before. (There are now six or more mixed clusters.) Various situations can arise pertaining to majorities with respect to the individual issues vs. majorities with respect to overall outcomes that are sufficiently surprising they have been referred to as *compound majority paradoxes*.[5]

First, once we have three issues, the majority cluster may be empty — that is, not only may the set of voters who prefer the majority options on all issues comprise less than a majority of all voters but it may be that no voter at all supports the majority option on all three issues. This is illustrated in Table 1, where the rows represent three issues *X*, *Y*, and *Z*, the columns represent five voters 1, 2, 3, 4, and 5, and the cells show the preference of each voter with respect to each issue, where '1' indicates preference for the majority position and '0' indicates preference for the minority position (so a majority of cells in each row contain a '1'). The majorities on issues *X* and *Y* are negatively correlated and therefore have minimal overlap, i.e., the single voter 3, but this voter supports the minority position on the third issue *Z*. Therefore, no voter supports the majority position on all three issues. This illustrates the *paradox of multiple elections* (Brams et al., 1998), i.e., if three or more two-sided issues are voted on simultaneously, it may be that the overall voting outcome, i.e., the majority-supported option on each issue, is not supported by any voter.

	1	2	3	4	5
X	1	1	1	0	0
Y	0	0	1	1	1
Z	1	0	0	1	1

Table 1

Table 2 illustrates several other compound majority paradoxes. The first is the *Anscombe paradox*: a majority of voters may be on the losing side of a majority of issues. In this case, voters 3, 4, and 5 (a majority) each lose on two issues out of three. Anscombe (1976) characterizes this as the 'frustration of the majority by fulfilment of the majority's will'. However, the paradox may seem less surprising once we recognize that many different majorities are involved: while a majority of voters (i.e., 3, 4, and 5) is frustrated on a majority of issues, each voter in that majority is frustrated on a different majority of issues (*Y* and *Z* for voter 3, *X* and *Z* for voter 4, and *X* and *Y* for voter 5); furthermore, the 'majority's will' on each issue that frustrates this majority is expressed by three different (though overlapping) majorities (1, 2, and 3 on issue *X*, 1, 2, and 4 on issue *Y*, and 1, 2, and 5 on issue *Z*).

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[5] These and similar paradoxes are discussed in more detail in Module 12.

	1	2	3	4	5
X	1	1	1	0	0
Y	1	1	0	1	0
Z	1	1	0	0	1

Table 2

A different interpretation of Table 2 leads to a perhaps odder paradox. Suppose that voters do not vote directly on issues *X*, *Y*, and *Z* (in the manner of 'direct democracy') but rather vote for either of two political parties that take opposing positions on issues, thus presenting voters with rival 'manifestos' or 'platforms' (in the manner of 'indirect', 'representative', or 'electoral democracy'). Let us also suppose that each voter cares more or less equally about each issue and therefore votes for whichever party supports the voter's preferred position on the greater number of issues. Finally suppose that party 1 supports the more popular (majority) position on every issue and party 0 supports the less popular (minority) position on every issue. We might therefore expect that the majority-pleasing party 1 would win the support of a majority of the electorate, but in fact party 0 wins the support of the majority 3, 4, and 5. This has been dubbed the *Ostrogorski paradox* by Rae and Daudt (1976).[6]

Finally, we can provide a third (perhaps) paradoxical interpretation of Table 2. In this case, the 'issues' *X*, *Y* and *Z* disappear and the five columns now represent five 'districts' or 'constituencies' each with three voters (so there are 15 voters in total) who (for unspecified reasons) vote for either party 1 or party 0. The party that wins a majority of districts (and therefore a majority of parliamentary seats) wins the election. While party 1 wins the first two districts, party 0 wins the other three districts and thus the election. This occurs despite the fact that party 1 has won a majority of the total vote (9 votes out of 15 or 60%). The problem is that party 1 carried a minority of districts with overwhelming support, while party 0 lesser support is spread more efficiently over a majority of districts. This phenomenon is referred to as the *referendum paradox* or an *election inversion* (or *reversal*) among other names.[7]

Bibliographical Notes and Further Readings

The distinction between Simple and Absolute Majority Rule is highlighted by Dougherty and Edward (2010). While May (1952) provides the most famous characterization of Simple Majority Rule, other rationales are proposed by Rae (1969), Taylor (1969), and Straffin (1977). Compound

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[6] The name refers to Moise Ostrogorski, a pre-revolutionary Russian political sociologist who was critical of political parties. Though we have used the same example to illustrate both the Anscombe and Ostrogorski paradoxes, they are logically independent, in that it is possible to construct examples that illustrate one but not the other.

[7] Real world examples include the 1951 U.K. general election and the 2000 and 2016 U.S. presidential elections (with respect to electoral votes rather than parliamentary seats). The phenomenon is further discussed in the next module.

majority paradoxes are discussed in Nurmi (1999, Chapter 7) and also see Module 12. The possible sources of a referendum paradox or 'election inversion' are identified, in the particular context of the U.S. Electoral College, in Miller (2012).

4. Majority Preference over Multiple Alternatives

To this point we have considered only two-sided issues. We now consider issues with multiple options, such as an election with three or more candidates or a bill in a legislative body together with one or more proposed amendments. We will also reconsider the case of two or more two-sided issues, which entails multiple possible outcomes.

4.1 Majority and Plurality Winners

Recall that a *preference profile* is a set of preference orderings (or ballot rankings) of alternatives, one for each voter. We will make many references to the eight (non-anonymous) profiles shown below for three voters (labelled 1, 2, and 3) and three or four alternatives (x, y, z, and v).

	Profile 1			***Profile 2***			***Profile 3***			***Profile 4***		
	1	2	3	1	2	3	1	2	3	1	2	3
first pref.	x	y	y	x	y	z	x	y	z	z	x	y
second pref.	z	x	z	y	x	y	y	z	x	y	z	x
third pref.	y	z	x	z	z	x	z	x	y	x	y	z

	Profile 5			***Profile 6***			***Profile 7***			***Profile 8***		
	1	2	3	1	2	3	1	2	3	1	2	3
first pref.	x	y	z	x	y	z	x	y	z	v	x	y
second pref.	v	v	v	y	z	x	y	z	v	z	v	x
third pref.	y	z	x	v	v	v	z	v	x	y	z	v
fourth pref.	z	x	y	z	x	y	v	x	y	x	y	z

An alternative that is ranked first by a majority of voters, such as y in Profile 1, is called a *majority winner*. Apart from the possibility of a tie, every ballot profile with just two alternatives has a majority winner, but when the number of alternatives is three or greater a majority winner may not exist, as in Profiles 2-8. However, there is always an alternative with the most first preferences, though two or more may be tied in this status; such an alternative is called the *plurality winner*. A majority winner, if one exists, is also the (unique) plurality winner. Note that the concepts of plurality and majority winner are defined on the basis of voter first-preferences only.

Note that Profiles 7 and 8 each include one *unanimous preference relationship*: for z over v in Profile 7 and for v over z in Profile 8. The *Pareto set* consists of all alternatives x such that at least one voter prefers x to every other alternative.[8] Thus the Pareto set is $\{x,y,z\}$ for Profile 7 and $\{x,y,v\}$ for Profile 8. For the other profiles shown above, the Pareto set includes the whole set of alternatives.

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[8] The concept is named after Vilfredo Pareto, an Italian economist of the late 19th and early 20th century, who developed the concept to assess the efficiency of economic allocations.

4.2 Condorcet Concepts

Given any profile and any pair of alternatives *x* and *y*, we can determine which alternative would win a majority in what the British call a 'straight fight', i.e., a strictly two-way contest. If a majority of ballots rank alternative *x* over alternative *y*, we say that '*x* beats *y*'. This defines the *pairwise simple majority preference relation* (which we shall henceforth refer to more simply as the *majority preference relation*) over alternatives, i.e., Simple Majority Rule applied to every pair of alternatives. This relation governs many political choice processes, including voting in parliamentary bodies (which typically entails a sequence of pairwise votes and is discussed in later sections of this module) and two-party electoral competition (in which each party chooses a 'manifesto' or 'platform' consisting of positions on each issue and voters choose between them, as discussed in Module 10).

The majority preference (or 'beating') relation is *asymmetric* (if *x* beats *y*, *y* cannot at the same time beat *x*) but may not be *complete* (since *x* and *y* may tie). A *Condorcet winner* is an alternative that beats every other alternative, while a *Condorcet loser* is an alternative that is beaten by every other alternative.[9] A majority winner must be a Condorcet winner, but the reverse is not true since, given three or more alternatives, a Condorcet winner need not be the first preference of a majority, or even a plurality, of voters — indeed, given four or more alternatives, a Condorcet winner need not be the first preference of any voter. These points are illustrated by Profiles 2 and 5, in which *y* and *v* are Condorcet winners. The set of alternatives that beat *x* is called the *win set* of *x* and is designated $W(x)$. Thus the win set of a Condorcet winner is empty, and a Condorcet winner belongs to the win set of every other alternative.

The distinction between a majority winner and Condorcet winner illustrates once again the point that multiple preference-based majorities may exist with respect to the same preference profile. On the one hand, if *x* is a majority winner, *x* beats every other alternative and does so through a single identifiable majority — namely, the majority consisting of the voters who have *x* at the top of their preference orderings, e.g., the majority of voters 2 and 3 in Profile 1. On the other hand, an alternative that is merely a Condorcet winner beats every other alternative but does so through different majorities. In Profile 2, *y* beats *x* through the majority of voters 2 and 3, while *y* beats *z* through the majority of voters 1 and 2. In Profile 5, *v* beats the other three alternatives through three different majorities (and no voter is common to all three majorities).

Though a Condorcet winner may exist when a majority winner does not, a Condorcet winner does not exist for every profile. This is illustrated by Profile 3 in Table 3, which produces a *Condorcet cycle* such that *x* beats *y*, *y* beats *z*, and *z* beats *x*, so there is neither a Condorcet winner nor a Condorcet loser. Such a cycle must entail three distinct majorities (here voters 1 and 3, 1 and 2, and 2 and 3). In Profile 4 each voter has a preference ordering that is the opposite of that in Profile 3, and it produces the opposite cycle such that *x* beats *z*, *z* beats *y*, and *y* beats *x*. Profiles 5 and 6 generate the same Condorcet cycle among *x*, *y*, and *z* as Profile 3, but in Profile 5 *v* is the Condorcet

[9] These names refer to the Marquis de Condorcet, the French enlightenment philosopher who proposed such a pairwise approach to social choice and voting and who discovered the cycling phenomenon discussed just below. For further details, see Module 2.

winner and there is no Condorcet loser, while in Profile 6 v is the Condorcet loser and there is no Condorcet winner. Profile 7 has a cycle that encompasses all four alternatives: x beats y beats z beats v beats x, while Profile 8 generates the opposite cycle. The Condorcet cycle phenomenon is also referred to as the *Condorcet paradox*, the *paradox of voting*, and *cyclical majorities*, among other names.[10]

4.3 Multiple Issues and Separable Preference

Armed with these Condorcet concepts, let us now reconsider aspects of the discussion in 3.3. The first point to make is that the earlier discussion took account only of the first preferences of voters with respect to each of the respective issues and not their full preference orderings over all possible outcomes. For example, three two-sided issues generate $2^3 = 8$ possible outcomes and, to identify Condorcet winners, losers, and cycles, we must take account of voters' full preference orderings over all of these outcomes. The second point to make is that, by specifying voter preferences on each issue separately, we implicitly assumed that voters have preferences that are *separable by issues* — that is to say, that each voter has a definite preference with respect to each issue *independent of how the other issues are decided.* Separability puts significant restrictions on admissible preference orderings.

To illustrate these points as simply as possible, let us further examine the case of the two-sided issues X_1 and X_2 discussed in 3.2. The complete set of alternatives (combinations of options, one for each issue) is (x_1,x_2), $(x_1,\bar{x}_2)$, $(\bar{x}_1,x_2)$, and $(\bar{x}_1,\bar{x}_2)$. While there are 4! = 24 possible strong orderings of four alternatives, only eight are consistent with separability. Suppose a voter prefers option x_1 to $\bar{x}_1$ on X_1, regardless of the outcome on X_2, and likewise prefers $\bar{x}_2$ to x_2 on X_2. Thus, the voter's most preferred alternative is $(x_1,\bar{x}_2)$ and his least preferred alternative is the opposite combination $(\bar{x}_1,x_2)$. But specification of the voter's preferences on each issue separately does not imply what his preferences are over all combined alternatives, for it is not apparent what the voter's preference might be between (x_1,x_2) and $(\bar{x}_1,\bar{x}_2)$. This choice poses this question to the voter: 'Which issue do you care about more?' or, more precisely, 'If you could get your way on just one of the two issues, which one would it be?'

Given a pair of two-sided issues, eight strong preference orderings are consistent with separability: each voter may prefer x_1 to $\bar{x}_1$ or vice versa, prefer x_2 to $\bar{x}_2$ or vice versa, and prefer to get his way on issue X_1 rather than issue X_2 (in the event he can get his way on only one) or vice versa. This leaves sixteen additional possible strong orderings of the four alternatives which are incompatible with separability. Similar considerations arise, though in a more complex fashion, with three or more two-sided issues.[11]

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[10] The singular term 'cyclical majority' is also used, but the term is inappropriate because the phenomenon necessarily involves three or more distinct majorities with no common member.

[11] Of course, issues may interact in such a way that voter preferences are *not* separable. For example, two proposed public works projects may be *complements* in that neither is of much value unless the other is authorized as well or they may be close *substitutes* in that, if one is built, there is little value in building the other also.

We can now consider the majority preference relation among the alternatives generated by multiple two-sided issues with separable preferences. Consider two alternatives that differ with respect to only one issue; clearly the alternatives that includes the majority-supported option on that one issue beats the alternative that includes the minority-supported option. Thus, in the two alternative case, (x_1,x_2) beats both $(x_1,\bar{x}_2)$ and $(\bar{x}_1,x_2)$, each of which in turn beats $(\bar{x}_1,\bar{x}_2)$. Generalizing, we have the following conclusion: *given multiple two-sided issues and voters with separable preferences, the only possible Condorcet winner is the alternative composed of the majority-supported option on every issue.*[12]

However, while only this alternative may be a Condorcet winner, it need not be. In the two issue case, only (x_1,x_2) can be a Condorcet winner, but it does not follow that (x_1,x_2) necessarily is a Condorcet winner. Since (x_1,x_2) differs from $(\bar{x}_1,\bar{x}_2)$ with respect to both issues, we cannot infer from separability that (x_1,x_2) beats $(\bar{x}_1,\bar{x}_2)$. In fact, $(\bar{x}_1,\bar{x}_2)$ may beat (x_1,x_2), as shown by the separable preferences in Profile 9.

Profile 9

	1	2	3
first pref.	(x_1,x_2)	$(x_1,\bar{x}_2)$	$(\bar{x}_1,x_2)$
second pref.	$(x_1,\bar{x}_2)$	$(\bar{x}_1,\bar{x}_2)$	$(\bar{x}_1,\bar{x}_2)$
third pref.	$(\bar{x}_1,x_2)$	(x_1,x_2)	(x_1,x_2)
fourth pref.	$(\bar{x}_1,\bar{x}_2)$	$(\bar{x}_1,x_2)$	$(x_1,\bar{x}_2)$

Note that, while voters 2 and 3 have opposed preferences on both issues, each cares more about the issue on which he is in the minority, so both prefer $(\bar{x}_1,\bar{x}_2)$ to (x_1,x_2) and $(\bar{x}_1,\bar{x}_2)$ therefore beats (x_1,x_2) through this 'coalition of minorities'. Thus, a party running on the platform $(\bar{x}_1,\bar{x}_2)$ would defeat a party running on the platform (x_1,x_2). Since there is no Condorcet winner, a cycle must be present. Indeed, a cycle encompasses all alternatives: (x_1,x_2) beats $(x_1,\bar{x}_2)$ beats $(\bar{x}_1,x_2)$ beats $(\bar{x}_1,\bar{x}_2)$ beats (x_1,x_2).[13]

This example may seem similar to that of the Ostrogorski paradox, but it differs in an essential way. In Table 2 (but using our present notation), $(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ beats (x_1,x_2,x_3) but not because voters 3, 4 and 5 care more about the issues on which they are in the minority but precisely because they care about all issues equally and thus all prefer $(\bar{x}_1,\bar{x}_2,\bar{x}_3)$, on which each voter gets his way on two issues, to (x_1,x_2,x_3), on which each gets his way on only one. The examples are similar, however, in that both imply the absence of a Condorcet winner and thus the presence of a Condorcet cycle.

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[12] The conclusion also generalizes to issues with multiple options. However, if any individual issue fails to have a 'majority-supported option' (i.e., a Condorcet winner with respect to the issue), there is no overall Condorcet winner either.

[13] If voter 1's preference between $(x_1,\bar{x}_2)$ and $(\bar{x}_1,x_2)$ were reversed, the latter would beat the former and the position of these two alternatives in the cycle would be reversed, but otherwise nothing would change.

Box 10: Directed Graphs, Tournaments, and Trees

A a *directed graph* consists of a set of *points* (also called *vertices* or *nodes*) together with *directed lines* (arrows) between pairs of points. Such a graph can represent a *binary relation* (Box X) over the set of points, and thus definitions pertaining to relations pertain also to directed graphs. A directed graph is *asymmetric* if, for any pair of points, there is at most one arrow between them. A directed graph is *complete* if there is at least one arrow between every pair of points. A directed graph is *transitive* if, whenever there is an arrow from point *x* to point *y* and from *y* to *z*, there is also an arrow from *x* to *z*. A directed graph that is complete, asymmetric, and transitive depicts a strict ordering of the points, but such a depiction is redundant; graphs are particularly useful for representing relations that are incomplete and/or intransitive.

A *path* is a sequence of distinct (non-repeating) points such that there is an arrow from each point to the following point; the *length* of a path is the number of arrows in it; point *y* is *reachable* from point *x* if there is a path from *x* to *y*. A directed graph is *strong* if every point is reachable from every other point, A *complete path* includes every point in the graph. A *cycle* is a path from *x* to *y* together with an arrow from *y* to *x*; the *length* of a cycle is the number of arrows in it, so the minimum length of a cycle is 3. A complete cycle includes every point in the graph.

A *tournament* is a directed graph that is both complete and asymmetric., so named because it might represent the results of a 'round-robin tournament' in which every contestant plays every other contestant exactly once in a contest that cannot end in a tie, where an arrow from *x* to *y* means that contestant *x* defeats contestant *y*. Figure 3 shows a strong tournament with seven points.

Here are several important properties of tournaments:

(1) every tournament has a complete path;
(2) a tournament is strong if and only if it has a complete cycle; and
(3) a strong tournament has a cycle of every length 3 through *m*, where *m* is the number of points.

Given a tournament *T*, a *subtournament* of *T* is a subset of the points in *T* and the arrows between points both of which are in that subset.

A *tree* is a directed graph in which there is (1) a unique *initial node* with no incoming arrows; and (2) at most one path from one node to any other. If there is an arrow from one node to another, we say the latter *follows* the former. This definition implies that (1) every node, other than the initial node, follows exactly one other node; (2) every other node is reachable from the initial node and this is true of no other node; (3) there is exactly one path from the initial node to any other node; and (4) there is a non-empty subset of nodes, called *terminal nodes*, from which no other node is reachable. A tree is *uniform* if every path from the initial node to a terminal node is of the same length. A *binary* tree has exactly two nodes following each non-terminal node. (See Figures 6-10 for examples of trees.)

4.4 Tournaments and Majority Voting

As the previous discussion suggests, given three or more alternatives we must consider not just the distribution of first preferences over alternatives but also the complete majority preference relation over all alternatives, so it is useful to have a compact way of representing this information. *Directed graphs*, as discussed in Box 10, provide such a device, where points represent alternatives and an arrow from x to y means that x beats y.[14]

However, much of the subsequent analysis is greatly simplified if we rule out the possibility of ties. This assumption can be justified in several ways. If we are considering large elections, the likelihood of a tie is very small. If we are considering small voting bodies, tie-breaking rules, such as those noted in Section 3.1, are often in place. But let us simply stipulate that the number n of voters is odd, every voter has strong preferences, and no voter ever abstains. This renders the majority preference relation complete, so that it is always represented by a *tournament* (see Box 10). All preference profiles we have considered meet these conditions, and the majority preference tournaments produced by Profiles 1-9 are depicted in Figure 1.

It is self-evident that, given any preference profile over m alternatives that does not entail ties, the majority preference relation can be represented by a tournament with m points. What is not self-evident is that every tournament can be generated by some preference profile. However, McGarvey (1953) demonstrated that this is in fact the case.

> ***McGarvey's Theorem***: Every tournament with m points represents the majority preference relation over m alternatives for some preference profile.

McGarvey demonstrated this 'constructively,' by showing how to construct a preference profile for which the majority preference relation is represented by any arbitrary tournament. Suppose a tournament with m points has an arrow from x to y. Then let the profile contain two voters who both prefer x to y but who have opposed preferences with respect to every other pair of alternatives — that is, one has the ordering $x, y, x_3, x_4, \ldots, x_{m-2}$ and the other $x_{m-2}, \ldots, x_4, x_3, x, y$. Let a similar pattern hold for every other pair of alternatives. This produces a profile with twice as many voters as there are pairs of alternatives, i.e., $2m(m-1)/2 = m(m-1)$ voters (see Box X on Combinations and Permutations), such that, with respect to each pair of alternatives, all but two voters have opposed preferences and the common preference of the two remaining voters determines majority preference in the required manner.[15]

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[14] However, the reader should be warned that some works use the reverse convention, so that an arrow from x to y means that x is beaten by y.

[15] If we stipulate an odd number of voters, we can add one additional voter (with any preferences); since every majority preference relationship carried by a margin of two, the additional voter does not change the tournament. Stearns (1959) showed that the required profiles can be constructed with no more than m+2 voters (m+1 if m is odd). McGarvey's theorem as stated can be generalized to say that any asymmetric but incomplete (i.e., allowing ties) directed graph can be generated by a profile of strong preference orderings.

Since many tournaments are cyclical, McGarvey's theorem also implies that majority preference may be cyclical, a point already directly demonstrated by Profiles 3-4 and 7-8. More generally, the theorem assures us that, if we examine all possible tournaments with *m* points, we are not studying any problem that cannot arise with respect to majority voting.

4.5 Tournament Solutions

What is the 'best', 'strongest', or 'most majority preferred' alternative in a tournament? The answer is clear if there is one alternative that beats every other alternative, i.e., a Condorcet winner, but such an alternative may not exist because of the cycling problem. A *tournament solution* provides an answer to this question that meets three conditions: (i) it exists for every tournament, (ii) it is neutral with respect to alternatives and (iii) it is the Condorcet winner when one exists. A number of tournament solutions have been proposed; here we discuss only the simplest ones that are especially relevant to the subsequent discussion.

The Copeland Set. Each alternative in a tournament can be assigned a (Copeland) *score* equal to the number of alternatives that it beats. A Condorcet winner has the highest possible score of $m-1$ and Condorcet loser has the lowest possible score of 0. A *Copeland winner* is an alternative with maximum score, so that no other alternative comes closer to being a Condorcet winner. But in the absence of a Condorcet winner, it may be that several (or many or all) alternatives have (equal) maximum scores, so we define the *Copeland set* as the set of all Copeland winners. In Figure 1, the Copeland sets in Tournaments 3, 4, and 6 are $\{x,y,z\}$ and in Tournaments 7 and 8 they are $\{x,y\}$.

The Top Cycle Set. Consider Figure 3, which displays a tournament with eight alternatives. (To make the diagram more readable, single arrows replace collections of arrows all pointing downward.) Note that there is no Condorcet winner (or Condorcet loser). However, every alternative in the set $\{x_1,...,x_4\}$ beats every alternative outside the set. While the same is true of $\{x_1,...,x_5\}$, $\{x_1,...,x_4\}$ is a subset of $\{x_1,...,x_5\}$ and no subset of $\{x_1,...,x_4\}$ has this property; since $\{x_1,...,x_4\}$ has a complete cycle, every alternative in the set is beaten by some other alternative in the set. This suggests the following definition: the *top cycle set* is the smallest set of alternatives such that every alternative in the set beats every alternative outside the set. Thus, the alternatives in the top cycle set collectively possess the property of a Condorcet winner with respect to alternatives outside the set. Indeed, a Condorcet winner, if it exists, meets the definition of the top cycle set (though the name is no longer entirely appropriate); at the other extreme, if the majority preference tournament has a complete cycle, the top cycle is the whole set of alternatives. In any event, a top cycle always exists and, in the absence of a Condorcet winner, includes at least three alternatives in a complete cycle. This implies that (i) every other alternative is reachable from any alternative in the top cycle, and (ii) there is a complete path beginning with every alternative in the top cycle set. It is clear that every alternative in the top cycle set has a higher Copeland score than any alternative outside the top cycle, so the Copeland set is a (not necessarily proper) subset of the top cycle. In Figure 1, the top cycle sets are respectively $\{y\}$, $\{y\}$, $\{x,y,z\}$, $\{x,y,z\}$, $\{v\}$, $\{x,y,z\}$, $\{x,y,z,v\}$, $\{x,y,z,v\}$, and $\{(x_1,x_2)$, $(x_1,\bar{x}_2)$, $(\bar{x}_1,x_2)$, $(\bar{x}_1,\bar{x}_2)\}$.

But we may note something paradoxical regarding Profiles 7 and 8. The voters in Profile 7 unanimously prefer *z* to *v*, so the Pareto set excludes *v*; and the voters in Profile 8 unanimously prefer *v* to *z*, so the Pareto set excludes *z*. Nevertheless, the top cycle sets include *z* and *v*. We now consider a tournament solution that is a subset of both the top cycle set and the Pareto set.

The Uncovered Set. Alternative *x covers y* if *x* beats *y* and *x* also beats every alternative that *y* beats. Thus in Tournament 7, *z* covers *v*, since *z* beats *v* and also the only alternative *v* beats, i.e., *x*, and likewise in Tournament 8 *v* covers *z*. (Note that this is true regardless of whether the Tournaments 7 and 8 are generated by preference profiles that entail unanimous preference.) If *x* covers *y*, $W(x)$, i.e., the set of all alternatives that beat *x*, is a proper subset of $W(y)$; it follows that *x* has a greater Copeland score than *y*. Since set inclusion is transitive, covering is a transitive (but incomplete, unless majority preference is itself transitive) subrelation (see Box X on Binary Relations) of majority preference. Moreover, suppose *x* is unanimously preferred to *y* and *y* beats *z*. Since all voters rank *x* above *y* and a majority of them rank *y* above *z*, the same majority (at least) rank *x* above *z*. Thus unanimous preference implies covering, even though covering does not require unanimous preference.

The *uncovered set*, i.e., the set of alternatives not covered by any other alternative, has a number of interesting properties. The transitivity of the covering relation implies that every tournament has an uncovered set and that every covered alternative is covered by an alternative in the uncovered set. There is a path of length no more than 2 from an uncovered alternative to every other alternative. If a tournament has a Condorcet winner, it is the unique uncovered alternative and covers every other alternative; likewise, every alternative in the top cycle covers every alternative outside the top cycle, so the uncovered set is a subset of the top cycle set. Since unanimous preference implies covering, the uncovered set is a subset of the Pareto set. An alternative with maximum Copeland score must be uncovered, so all Copeland winners belong to the uncovered set. Finally, Tournaments 7 and 8 in Figure 1, as well as the tournament in Figure 2, show that it may be a proper subset. In the absence of a Condorcet winner, the uncovered set includes at least three alternatives in a cycle.

Bibliographical Notes and Further Readings

The discussion on Section 4.3 is drawn from Miller (1975), but the basic result was established by many researchers more or less simultaneously and has since been greatly generalized by Schwartz (1981). Textbook discussions of tournaments are presented by Harary et al. (1965, Chapter 11) and Moon (1968). Early applications to majority rule and voting were presented by McGarvey (1953), Taylor (1968), Miller (1977 and 1980). Moulin (1986) and Laslier (1997) provide foundational discussions of tournament solutions, while Moser (2015) provides a recent introductory summary. Copeland (1951) proposed the Copeland score (but the only record of this paper's existence appears to be a citation and summary in Luce and Raiffa, 1957, p. 358). The concept of top cycle set was first explicitly propounded by Ward (1963), though it is implicit in Black (1958). The uncovered set was proposed and named by Miller (1980), though there were antecedents in earlier social choice literature.

5. Conditions for Condorcet Cycles

We have seen that some profiles produce Condorcet cycles while others do not, so it is natural to ask what conditions on preference (or ballot) profiles permit or preclude the phenomenon. To simplify matters, we continue to assume that all voters have strong preferences and that the number of voters is odd, so that no ties exist in majority preference.

The third property of tournaments noted in Box 10 implies that, if a tournament has a cycle of any length, it includes at least one cycle of length 3. Thus a condition on profiles that precludes cycles among any three alternatives also precludes cycles of greater length. Accordingly we can focus on the three-alternative case.

Given three alternatives *x*, *y*, and *z*, there are six possible strong preference orderings, as shown in Table 3. These orderings can be paired off, so the orderings in each pair are the opposite of each other. Let *X1*, *X2*, *Y1*, etc., label each ordering in terms of its second-ranked alternative and then subdivided into Type 1 and Type 2 orderings as shown in Table 3, and let *n*(*X1*) be the popularity of ordering *X1*, i.e., the number of voters with this ordering, and likewise for the other orderings.

	Y1	*Z2*	*X2*	*Z1*	*X1*	*Y2*
first pref.	*x*	*x*	*y*	*y*	*z*	*z*
second pref.	*y*	*z*	*x*	*z*	*x*	*y*
third pref.	*z*	*y*	*z*	*x*	*y*	*x*

Table 3

The logic of this labelling is reinforced by the *representation triangle* shown in Figure 4, which provides a convenient graphical representation of all three-alternative profiles.[16] An equilateral triangle with vertices (corner points) representing alternatives *x*, *y*, and *z*, can be divided into six regions according to the distance of points within them from each vertex. For example, all the points in the northwest region in Figure 1 are closest to *x*, second closest to *y*, and furthest from *z*; thus, this region corresponds to the first preference ordering in Table 3. In terms of the triangle, we can characterize some pairs of ordering as *opposite*, e.g., *X1* and *X2*, and others as *adjacent*, e.g., *Y1* and *Z2*.

Two voters with opposite preference orderings share the same second preference but disagree on every choice between pairs of alternatives. Two voters with adjacent orderings share either the same first preference or the same third preference, and therefore agree on two out of three choices between pairs of alternatives. Two voters with different orderings of the same type do not rank any alternative in the same position but agree on one choice between pairs of alternatives.

Given Simple Majority Rule with ties precluded, every preference profile produces one of eight majority preference patterns: one of the six *orderings* of three alternatives shown in Table 3, or one of two cyclical patterns, i.e., the *forward cycle* such that *x* beats *y* beats *z* beats *x* results or the *backward cycle x* such that beats *z* beats *y* beats *x*. Two types of conditions on strong preference profiles over three alternatives are sufficient to preclude Condorcet cycles.

16 The uses of geometric representations of profiles are discussed in more detail in Module 14.

5.1 Popularity Conditions

Popularity conditions pertain to the relative popularity of preference orderings. *Strong popularity dominance* holds if a majority of voters have the same preference ordering; clearly majority preference is the same ordering, so a cycle is precluded regardless of the distribution of voters over the other orderings. *Weak popularity dominance* holds if a majority of voters have adjacent preference orderings, so that they share either a first or third preference; this assures that there is a Condorcet winner or loser, so a cycle is precluded regardless of the distribution of voters over other orderings.

5.2 Exclusion Conditions

In contrast to popularity conditions, *exclusion conditions* stipulate that certain orderings are excluded, i.e., have zero popularity, but stipulate nothing about the relative popularity of the non-excluded orderings.

First, we note that, if all voters have Type 1 (or Type 2) orderings and (strong) popularity dominance does not hold, a *forward cycle* (or a *backward cycle*) results. More generally, a Condorcet cycle can result only if a profile includes all three Type 1 or all three Type 2 orderings. To see this, note that every (cyclical or non-cyclical) majority preference pattern must include one alternative that beats one alternative and is beaten by the other, e.g., some y such that x beats y beats z. Since x beats y if and only if

$$n(X1) + n(Z2) + n(Y1) > n(X2) + n(Z1) + n(Y2)$$

and y beats z if and only if

$$n(Y1) + n(X2) + n(Z1) > n(Y2) + n(X1) + n(Z2),$$

we can add the two inequalities and simplify to tell us that $n(Y1) > n(Y2)$. Since $n(Y2) \geq 0$, it follows that $n(Y1) > 0$. Generalizing, the forward cycle can occur only if $n(X1) > 0$ and $n(Y1) > 0$ and $n(Z1) > 0$, and the backward cycle can occur only if $n(X2) > 0$ and $n(Y2) > 0$ and $n(Z2) > 0$.

Putting the matter the other way around, a voting cycle is precluded if at least one ordering of each type is excluded. This means that either two adjacent or two opposite orderings are excluded. In the event two adjacent ordering are excluded, either (1) there is some alternative that no voter ranks third or (2) there is some alternative that no voter ranks first. In the event of (1), the alternative that no one ranks third beats both other alternatives unless popularity dominance condition comes into play, so in any case a cycle is precluded. In the event of (2), only two alternatives have any first preferences, so one is a majority winner and a cycle is precluded. In the event that two opposite orderings are excluded, (3) there is some alternative that no voter ranks second and two pairs of adjacent orderings remain, and the orderings in one pair must be held by a majority of voters, so weak population dominance holds and a cycle in precluded.[17]

[17] Since these conditions preclude some alternative from being ranked first, second, or third by any voters, Sen (1966) called them *value-restriction* conditions.

Each of these conditions can be interpreted in a way that gives it plausibility in some circumstances. Condition (1) would hold if the three alternatives are commonly perceived, respectively, as 'left-wing' (or relatively 'extreme' in some other sense), 'right-wing' (or relatively 'extreme' in the opposite sense), and 'centrist' (i.e., a compromise between the other two); the 'centrist' alternative would presumably be the second preference of every voter who has either 'extreme' alternative as a first preference. Condition (2) would hold if two alternatives are commonly perceived to be nearly indistinguishable 'clones' of each other such that no voter ranks the relatively 'distinctive' third alternative between them. Condition (3) would hold if, for example, voters live along a road, the alternatives pertain to the location of some 'public bad' (e.g., a garbage dump) that all voters want to have located as far away from their homes as possible, and the three alternative locations are at opposite ends of the road and somewhere in between.

5.3 Net Preference Profiles

While each condition previously identified is *sufficient* by itself to preclude a cycle, many profiles that meet none of these conditions fail to produce a cycle. Indeed, if there are many voters with diverse preferences, it is almost certain that no exclusion condition holds and rather unlikely that weak popularity dominance holds. Yet many — indeed most — such profiles do not produce cycles. We now state conditions that are necessary as well as sufficient to preclude a cycle among three alternatives.

Clearly many different preference profiles may generate the same majority preference pattern. In particular, if we have a preference profile and then add two voters with opposite preference orderings, the pattern is unchanged because the two new voters 'cancel out' in each pairwise majority preference. Likewise, if two voters with opposite preferences are removed from a profile, majority preference is unchanged.

Generalizing the latter consideration, we can take any preference profile over three alternatives and 'reduce' it by removing pairs of voters with opposite preferences until the profile includes no voters with opposed preferences. This leaves us with a *net preference profile* in which no more than three of the six possible orderings have positive popularity but which produces the same majority preference tournament as the original profile; since the three orderings cannot include opposites, they must be either (i) all Type 1, (ii) all Type 2, or (iii) adjacent.[18] In the event of (i) or (ii), the net (and therefore the original) profile produces a (forward or backward) cycle unless (strong) popularity dominance holds for the net profile; in the event of (iii), the net (and therefore the original) profile does not produce a cycle.

5.4. Single-Peaked Preferences

We have identified exclusion conditions that are sufficient to preclude cycles in a set of the three alternatives. In order to preclude cycles in a larger set of alternatives, it is not necessary for the *same* condition to hold for all triples of alternatives. However, the generalization of exclusion

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[18] In the event of (iii), exclusion conditions (1) and (2) both hold in the net profile; however, condition (3) cannot hold in a net profile.

condition (1) to the whole (perhaps infinite) set of alternatives is especially relevant politically. The notion is that all alternatives can be located along a line (i.e., a one-dimensional 'space') representing a policy or ideological dimension and that voter preferences are shaped accordingly. It is natural to assume that each voter *i* has a point of maximum preference (often called an *ideal point)* and that each voter's preferences among alternatives relate to their distance from his ideal point. A standard assumption is that voters have what Black (1948, 1958) called *single-peaked* preferences: given two alternatives *x* and *y* that lie on the same side of his ideal point, a voter prefers the closer one. (However, if *x* and *y* lie on opposite sides of his ideal point, the voter may have either preference (or be indifferent) between *x* and *y*.)

Figure 5 illustrates why such preferences are said to be 'single-peaked'. The horizontal axis is the line over which alternatives are located (*y*, *z*, *v*, and *w* being explicitly shown), together with five voter ideal points labelled x^1, x^2, etc. The vertical axis represents the voters' relative preference for alternatives. The preference graph of each voter is literally single-peaked, rising without interruption until it reaches a (single) peak at the voter's ideal point and declining without interruption thereafter. Note that single-peaked preferences may be asymmetric about an ideal point, as for voter 1. The preference orderings over *y*, *z*, *v*, and *w* of the five voters are shown.

Black demonstrated that single-peaked preferences preclude Condorcet cycles. Moreover, if the number of voters is odd, the Condorcet winner is the most preferred alternative of the *median voter*, i.e., the voter who has fewer than half of the other voters' ideal points to his left and fewer than half to his right.[19] This result is commonly referred to as the *Median Voter Theorem*. In Figure 5, voter 3 is the median voter and *v* is 3's most preferred alternative in the set {*y*, *z*, *v*, *w*}, so *v* is the Condorcet winner in that set, while x^3 is the Condorcet winner in the set of all possible alternatives along the line.

A natural generalization of single-peakedness can extend the notion from the single dimension case to the case of two or more dimensions while maintaining the spirit of single-peakedness, i.e., that voters generally prefer alternatives that are closer to their ideal point to those that are more distant. However, given more than one dimension, generalized single-peakedness does not assure a Condorcet winner or preclude cycles.

5.5 Distributive Preferences

Having focused on conditions that preclude Condorcet cycles, we should also note that other conditions, such as those discussed in 4.3, make Condorcet cycles likely. Indeed, one politically relevant condition on preference profiles produces a maximally cyclical majority preference relation. While single-peakedness may pertain to varying levels of provision of some *public good* (and associated costs), sometimes alternatives are, either explicitly or in effect, proposed allocations among the voters (or, if the voters are representatives, among their constituencies, which tends to induce similar preferences among legislators motivated to please their constituents) of some divisible and essentially *private good*, i.e., they are different ways of parceling out among voters (or their constituencies) particularized benefits (such as public works projects) the costs of which are shared

19 Two or more median voters may share the same ideal point. If the number of voters is even, there may be no Condorcet winner but at least one alternative is unbeaten.

by all, and each voter seeks to maximize his share of the good. This pattern of political decision making has been called ‘pork-barrel politics’ in the American context, and the term ‘distributive politics’ is widely used in academic political science. Clearly this political context structures preference profiles in a distinctive manner but not in the ‘ideological’ manner suggested by single peakedness.

The simplest formulation of distributive preferences is as follows. Each alternative corresponds to some allocation of a divisible good among voters. Such an alternative q may be described by a list of (non-negative) numbers $q = \langle q_1,q_2,...,q_n \rangle$, where each q_i is the share of the good received by voter i and the q_i's add up to the total amount of the good. If voters seek to maximize their shares, voter i prefers q to q' if and only if $q_i > q'_i$. In the context of pure allocation, majority preference is maximally cyclical in this sense: for every pair of alternatives q and q' such that q' beats q, there is a third alternative q'' such that q'' beats q' but q beats q''. Put otherwise, every pair of alternatives belongs some cyclic triple.

To see this, let the set of voters be partitioned into three sets I, J, and K such that none contains a majority of voters. Consider an allocation q and a second allocation q' that beats q, as it benefits all voters in I and J at the expense of all those in K. Then a third allocation q'' can readily be designed that beats q' by favoring all voters in J and K at the expense of those in I and at the same time is beaten by q, producing a cycle among the three allocations. Table 4 provides a simple example in which I, J, and K are one-voter sets.

Allocation	*I*	*J*	*K*	Total
q	3	3	4	10
q'	4	4	2	10
q''	2	5	3	10

Table 4

5.6. Prevalence of Condorcet Cycles

It is natural to ask how prevalent Condorcet cycles are, particularly those that preclude a Condorcet winner? One way to address this question is to calculate the proportion of profiles that fail to produce a Condorcet winner. The simplest case is that of three voters, each of whom has one of the six possible strong preference orderings over three alternatives shown in Table 3. Altogether there are $6^3 = 216$ possible (non-anonymous) profiles, four of which are Profiles 1-4. Profile 3 produces a cycle, as do the five other profiles composed of the same three orderings but different assignments of a single voter to each ordering. In addition, Profile 4 is one of six profiles that produce the reverse cycle. Thus, $12/216 \simeq 0.0556$ of all profiles with three voters and three alternatives fail to produce a Condorcet winner. More burdensome counting operations show that the proportion of profiles that fail to produce a Condorcet winner increases somewhat as the (odd) number of voters increases, reaching about 0.08 when $n = 11$, and approaches a limit of about 0.0877 when the number of voters becomes very large. Still more burdensome calculations show that the proportion of profiles that fail to produce a Condorcet winner increases much more rapidly when the number of alternatives increases.[20] Given a very large number of voters, the proportion increases from about 0.0877 with the three alternatives to about 0.5 with 11 alternatives and then to about 0.85 with 50 alternatives, and it approaches 1 as the number of alternatives becomes very large.

If we assume voters select their preference ordering randomly — for example, by flipping coins if there are two alternatives (and two possible orderings), rolling dice if there are three alternatives (and six possible orderings), and so forth — these proportions become probabilities. The circumstance in which voters randomly select orderings has been called an *impartial culture*, though it might better be characterized as the absence of any 'culture' at all — that is, the absence of any characteristic structuring of public opinion. Given a large number of voters, the impartial culture implies that every pairwise vote between two alternatives is a virtual tie and thus that every majority preference tournament is essentially equally likely. Since, as the number of alternatives increases the number of distinct tournaments increases much more rapidly than the number of tournaments with a Condorcet winner, the probability that there is a Condorcet winner essentially vanishes when the number of alternatives becomes very large.[21] Indeed, when the number of alternatives becomes very large, almost all tournaments are maximally cyclic (as in the distributive case), which implies that both the top cycle set and the uncovered set almost always equal the whole set of alternatives.

However, the impartial culture assumption is in effect a worst-case analysis. For example, even a slight tendency toward single-peakedness greatly reduces the likelihood of cycles. Moreover, even if the majority preference tournament over some very large set of 'potential' alternatives is highly cyclic, it is quite possible and perhaps likely that a majority preference over the much smaller set of alternatives that may be actually 'on the agenda' for voting (as candidates in an election or motions before a parliamentary body) does not include a cycle.

In point of fact, very few clear empirical examples of cyclical majorities have been found. In large part, this is because appropriate data, i.e., preference *orderings* over three or more alternatives, is rarely available. Election results and public opinion polls typically reveal only information about the distribution of first preferences and, as we shall see in the following sections, parliamentary voting does not provide enough preference information to reveal cycles. In the few cases in which appropriate data is available — for example, individual ballots under a voting rule that requires voters to rank candidates — voting cycles are occasionally found but they appear to be quite exceptional and, even when they exist, are rarely top cycles.

Still, in so far as alternatives have a distributive component, or alternatives consist of distinct issues, e.g., public works projects, such that no single one is majority-supported but larger 'packages' are majority-supported (as discussed in 4.3), cycles appear to be likely or even inevitable on

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[20] Recall that for $m > 3$, there is a distinction between the existence of a cycle and the absence of a Condorcet winner, as there may be a non-top cycle such as that produced by Profile 5. The proportions given here pertain to the absence of a Condorcet winner.

[21] This probability, which is 0.75 with three alternative, falls to about 0.01 with 11 alternatives and 0.00004 with 20 alternatives.

theoretical grounds. Moreover, we can describe other plausible scenarios based on real-world politics that entail cycles. U.S. politics in the 1950s pertaining to the 'Powell Amendment' provides a classic example (that we will refer to again later). Representative Adam Clayton Powell (from the Harlem district in New York City) regularly proposed amendments prohibiting racial discrimination to legislation authorizing the distribution of federal funds to states. Most notably, he proposed to attach his amendment to Democratic legislation, generally opposed by Republicans, that would have provided federal aid to states for education. Northern Democrats generally favored the amendment, as did most Republicans (who were still 'the party of Lincoln'), but segregationist Southern Democrats vehemently opposed it. Thus the Congressional preference profile looked like this, where *b* is the federal aid bill, *a* is the Powell amendment, and *q* is the status quo (i.e., no federal aid):

Profile 10

ND	SD	R
$b + a$	b	q
b	q	$b + a$
q	$b + a$	b

Since none of the three blocs was of majority size, the profile produced a cyclical majority.

Bibliographical Notes and Further Readings

The literature surveyed in this section is vast. The three-alternative exclusion conditions were identified by Sen (1966). The notion of net preferences is adapted from Feld and Grofman (1986 and 1988); also see Niemi (1969). Black (1948 and 1958) made explicit (and named) the concept of single-peaked preferences, and thereby introduced the 'spatial model of voting'; Miller (2015) provides an introductory summary. Ward (1963) provides an early discussion of distributive preferences and their consequences for majority cycling; Epstein (1998) provides a more recent one. Gerhlein (2006) and Gerhlein and Lepelley (2011) comprehensively summarize the vast literature on the conditions for and prevalence of Condorcet cycles. Fey (2008) shows that almost all very large tournaments are maximally cyclic. Riker (1982 and 1986) claims to identify empirical examples of Condorcet cycles (including the Powell amendment), though Mackie (2003) disputes many of his claims. Feld and Grofman (1992) examine a unique data set to look for Condorcet cycles and find almost none.

6. Voting Rules for Multiple Alternatives

We have taken note of several voting rules to choose between two alternatives and have discussed various concepts — majority winner, plurality winner, Condorcet winner, etc. — that suggest possible approaches for choosing among multiple alternatives. We now consider a variety of voting rules for choosing one alternative from among multiple alternatives. Most plausibly these may be thought of as rules for electing a single candidate out of a field of three or more candidates. These rules all satisfy May's Anonymity and Neutrality conditions — that is, they treat all voters and candidates interchangeably. Some are commonly used in public elections; others are used in private elections (e.g., of scholarly societies) or informal committees (e.g., faculty meetings); a few are of theoretical interest only. We defer consideration of voting rules for voting by representatives, i.e., rules of the parliamentary type, where the alternatives are motions, bills, amendments, etc., and to Modules 7 and 13 consideration of voting rules for selecting several candidates out of a larger field.

6.1 Majoritarian Voting Rules

Given this module's focus on majority voting, we restrict our attention to voting rules that are in some sense 'majoritarian'. But we can attribute a number of distinct and increasingly demanding meanings to this term. First, a voting rule is *weakly majoritarian* if it reduces to Simple Majority Rule in the special case in which there are just two alternatives. Virtually all anonymous and neutral voting rules, including all those that we will discuss here, are weakly majoritarian. Second, a voting rule is *strictly majoritarian* if it gives any majority *coalition*, i.e., any set of voters acting in concert, the power to determine the winner of the election. This means that, if the members of such a coalition agree on an alternative *x* that they want as the outcome, they can coordinate their votes (e.g., their ballot rankings) in such a way that *x* becomes the voting outcome *regardless of what the remaining voters do*. Most but not all rules we consider here are strictly majoritarian. Third, a voting rule is *simply majoritarian* if all that members of a majority coalition must do to assure that alternative *x* is the winning alternative is to place *x* at the top of all of their ballot rankings, i.e., such a rule guarantees that a majority winner becomes the winning alternative. Finally, a voting rule is *Condorcet consistent* if, given that there is a Condorcet winner *x* with respect to the ballot profile, *x* is the winning alternative. Many voting rules described in this section fail to be majoritarian in this strongest sense.

6.2 Two versus Three or More Candidates

May's Theorem tells us that, when there are just two candidates, Simple Majority Rule uniquely meets four desirable conditions. It is also (trivially) majoritarian in every sense defined above. But once the number of candidates expands to three or more, additional conditions become relevant and a variety of problems arise. First, many different apparently fair and reasonable voting rules (including those discussed below, along with other more esoteric possibilities) are available (and quite a few are in actual use). While each rule reduces to simple SMR in the two-candidate case, different rules often select different winners in the multi-candidate case. Moreover, all such voting rules have evident flaws. Indeed, we can identify two important flaws that are essentially unavoidable when choosing among three or more candidates (or other alternatives).

First, given a 'straight fight' between just two candidates, no voter ever has reason to consider voting other than 'sincerely' for his more preferred candidate. Put more formally, given just two alternatives Simple Majority Rule (or any other non-negatively responsive rule) is *strategyproof*, in that no voter can ever improve the outcome with respect to his preferences by misreporting those preferences on the ballot. But no (deterministic) voting rule whatsoever is strategyproof given three or more candidates.[22]

Second, all voting rules are vulnerable to *spoiler effects* when the field of candidates expands or contracts — that is, whether candidate *x* or *y* is elected may depend on whether some third candidate *z* (the potential 'spoiler') enters, or exits from, the field.

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[22] 'Non-deterministic' voting rules that entail lotteries may be strategyproof, regardless of the number of candidates — for example, a rule under which all voters submit ballot rankings, one ballot is selected at random, and the candidate at the top of that ranking is elected. This problem is discussed in much more detail in Module 16.

In the following discussion, we will refer to the following (anonymous) Profile 11 with 100 voters. To make the discussion more concrete, we use the traditional British party labels to identify three candidates — Labour, Liberal, and Conservative — one of whom is to be elected. While there are six possible strong orderings of three candidates, we first consider this simple profile in which only three orderings are present. The numbers indicate the popularity of each ordering.

Profile 11

# of voters =>	46	20	34
first pref.	*Labour*	*Liberal*	*Conservative*
second pref.	*Liberal*	*Conservative*	*Liberal*
third pref.	*Conservative*	*Labour*	*Labour*

Note that this profile has no majority winner; Labour is the plurality winner, Liberal is the Condorcet winner and, despite being the plurality winner, Labour is the Condorcet loser.

6.3 Plurality Rule

Under *Plurality Rule*, such as is used in British parliamentary elections (and called 'First-Past-The-Post') and most U.S. elections, each voter votes for exactly one candidate, and the candidate receiving the most votes wins. We assume that Profile 11 is both a preference and ballot profile — that is to say, all voters vote sincerely. Plurality Rule takes account of only the first preferences, so the *plurality ranking* is as follows:

Candidate	Votes Received (= First Preferences)
Labour	46 votes (*winner*)
Conservative	34 votes
Liberal	20 votes

Plurality Rule elects the *plurality winner* as previously defined.

In effect, Plurality Rule asks each voter to imagine that he or she is a 'dictator' who can unilaterally determine which candidate is to be elected (so there is no need to rank all candidates). Given that different voters may select different candidates, the reasonable thing to do is to elect the candidate selected by the most voters.

6.4 Borda Rule

Evidently, there were few objections to using plurality rule in multi-alternative elections until the late 18th century, when the French engineer and mathematician Jean-Charles de Borda pointed out its various flaws (discussed in the next section) and proposed his own 'method of marks' — now generally called the *Borda Rule* — to the (French) Academy of Sciences.[23] Borda argued that voters should rank all the candidates and that a proper voting rule should take account of these full rankings. Specifically, he proposed that, given *m* candidates, a candidate should be awarded zero points (or 'marks') for each ballot on which he is ranked last, one point for each ballot on which he is ranked

[23] See Module 2 on the history of voting.

second-to-last, and so forth, until being awarded $m - 1$ points for each ballot on which he is ranked first. The points for each candidate are then added up and the candidate with the highest total *Borda score* is elected.[24] Here are the Borda scores and the *Borda ranking* for Profile 11:

Candidate	Borda Score
Liberal	120 (*winner*)
Labour	92
Conservative	88

Labour no longer wins because the Borda Rule takes account of the fact that, although Labour has the most first preferences, Labour is also ranked last by all other voters; and Liberal benefits from the fact that, although Liberal has the fewest first preferences, no voters rank Liberal last.

6.5 Condorcet Voting

Borda's contemporary the Marquis de Condorcet responded favorably to Borda's critique of Plurality Rule and endorsed his suggestion that voters should rank all the candidates, but Condorcet identified problems (discussed in the next section) with Borda's method of aggregating these rankings. Condorcet proposed instead that elections should be based on what we have previously called the *majority preference relation*. That is, we look at all possible *pairs* of candidates and see which candidate in each pair is supported by a majority of voters. For Profile 11, we get the following 'Condorcet ranking':

Pairwise vote	Pairwise winner	Condorcet Ranking
Liberal vs. *Conservative*:	*Liberal* wins (66-34)	*Liberal* (Condorcet winner)
Conservative vs. *Labour*:	*Conservative* (54-46)	*Conservative*
Liberal vs. *Labour*:	*Liberal* wins (54-46)	*Labour* (Condorcet loser)

We may note that this Condorcet ranking is precisely the *opposite* of the plurality ranking based on first preferences only and that it also differs from the Borda ranking that is also based on full orderings.

An interesting connection between the Borda Rule and Condorcet's approach is illustrated in the matrix displayed in Table 5. The matrix has three rows and three columns labelled by the names of the candidates. Each cell of the matrix (other than those on the diagonal that pairs each candidate with himself) shows the number of votes won by the row candidate when paired against the column candidate. Note the sum of entries in each row is equal to the candidate's Borda score, as shown in the next column. The final column shows the number of pairwise wins for each candidate. The source of the reversal of the positions of Conservative and Labour in the Borda and

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[24] Any equally spaced system of scores produces the same Borda ranking and winner, but the simple system described in the text produces the equivalence displayed in Tables 5 and 6. The Borda rule belongs to the broader class of 'scoring rules'. In general, a *scoring rule* awards s_i points to a candidate for each ballot on which the candidate is ranked in the i-th position, where $s_1 \geq s_2 \geq \ldots s_m$ and $s_1 > s_m$. Plurality rule is a scoring rule with $s_1 = 1$ and all other $s_k = 0$.

Condorcet rankings now becomes clear. Labour narrowly loses to both other candidates, while Conservative narrowly beats Labour but loses badly to Liberal; thus while Labour never wins and is the Condorcet loser, Labour collects more total votes (Borda points) than Conservative.[25]

	Lib	*Cons*	*Lab*	score	wins
Lib	—	66	54	120	2
Cons	34	—	54	88	1
Lab	46	46	—	92	0

Table 5

It is tempting to conclude that we can identify a 'Condorcet rule', under which voters rank all candidates and the Condorcet winner is elected. But as we have seen, Condorcet also discovered the phenomenon of cycles, which implies that a Condorcet ranking (and winner) may not exist, as is illustrated by Profile 12.

Profile 12

# of voters =>	46	20	34
first pref.	*Labour*	*Liberal*	*Conservative*
second pref.	*Liberal*	*Conservative*	*Labour*
third pref.	*Conservative*	*Labour*	*Liberal*

Notice that first preferences in Profile 12 are unchanged from Profile 11, so it remains true that there is no majority winner and Labour is still the plurality winner. However, Labour is now the Borda winner as well. But the crucial difference is that there is now a cycle among the three candidates, as is apparent in Table 6.[26]

	Lib	*Cons*	*Lab*	score	wins
Lib	—	66	20	86	1
Cons	34	—	54	88	1
Lab	80	46	—	126	1

Table 6

Condorcet's own discussion of what to do in the event there is no Condorcet winner is notably ambiguous and open to different interpretations. But given Table 4, Condorcet might have suggested that the 'proposition' that 'Conservative beats Labour' is the weakest link in the cycle,

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25 This phenomenon bears some resemblance to the 'referendum paradox' discussed in 3.3.

26 Note that Profile 11 is single-peaked (in a natural manner, with Labour as 'leftwing' and Conservative as 'rightwing' with Liberal 'between' them), while Profile 12 is not value restricted (as the Conservative voters, perhaps implausibly, prefer Labour to Liberal).

since the relationship would be reversed if only five voters were to reverse their preference for Conservative over Labour, whereas reversing either of the other two 'propositions' would require a greater number of individual reversals.

The Borda rule is not commonly used in public elections. In fact, most voting rules for electing a single candidate in actual use are variants of plurality rule, perhaps with some kind of runoff.

6.6 Plurality Runoff Rules

In the event that Plurality Rule does not give one candidate an absolute majority of votes, *Plurality Runoff Rule* prescribes a runoff election between the top two candidates in the plurality ranking. Thus Profiles 11 and 12 would both produce runoffs between Labour and Conservative, which Conservative wins in Profile 11 and Labour wins in Profile 12. Plurality Runoff Rule is quite common, being used in French (and many Latin American) presidential elections and some US elections. Usually a week or two elapses between the first and second round of voting.

If there are four or more candidates, a more elaborate *Multi-Round Plurality Runoff Rule* can be used, under which candidates are eliminated one at a time and balloting continues until a majority winner exists among the surviving candidates. Such multi-round voting may be feasible in an assembly (for example, to select a representative to a higher-level assembly) but not in a large electorate (that must be called back to the polls multiple times).

6.7 The Alternative Vote

The second trip to the polls that may be required by Plurality Runoff Rule can be avoided if voters initially rank the candidates on a single ballot; if there is no majority winner, ballots cast for non-surviving candidates are transferred to the higher ranked of the top two candidates in the plurality ranking.[27]

But if voters are asked to rank all candidates, it makes sense to eliminate candidates one at a time. The *Alternative Vote* (AV) — also known (especially in the U.S.) as *Instant Runoff Voting* — is multi-round plurality runoff rule made feasible for mass elections by having voters rank all candidates on a single ballot.[28] If a majority winner now exists, that candidate is elected. Otherwise the candidate with the fewest first preferences is eliminated and his or her ballots are transferred to surviving candidates on the basis of second preferences. If a majority winner with respect to original and transferred ballots exists, that candidate is elected. Otherwise, the remaining candidate with the

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[27] Plurality Runoff Rule using ranked ballots rather than an actual runoff election is sometimes called the *Contingent Vote*. A weak version in which voters can express only first and second preferences, which means that not all ballots are transferable, is called the *Supplementary Vote* and has been used recently in some local elections in England.

[28] AV/IRV is also known as the *Hare Rule* as well as by other names, and it is equivalent to the Single Transferable Vote (discussed in Module 7) applied to the case in which a single candidate is to be elected. AV is used to elect members of the Australian House of Representatives but its use to elect members of the U.K. House of Commons was rejected by British voters in a 2011 referendum.

fewest first preferences is eliminated and his or her ballots are transferred on the basis of second or third preferences. And so forth, until there is a majority winner in the reduced field of candidates (which, in any case, will be the case when only two candidates survive).

It should be evident that, if there are more than three candidates, the final pairwise 'runoff' under AV may be between different candidates than those in the single runoff under Plurality Runoff Rule, and that the rules may therefore elect different candidates. Table 7 shows the candidates (and their approximate ideological positions) in the 2002 French presidential election, together with the results of the first and second round votes. The second-round runoff unexpectedly was between Chirac and Le Pen, because the many minor left-wing candidates in the field drew first-preference support away from Jospin and drove him into third place in the plurality ranking. However, if AV had been in use, the minor candidates would have been eliminated one by one and most of their ballots would (assuming voter preferences are approximately single-peaked) have transferred to Jospin and certainly very few would have transferred to Le Pen. As a result, Jospin would have been in the final 'runoff' with Chirac and, depending on how supporters of minor centrist candidates voted, might have won.

Candidate	1st Round	2nd Round
Jacques Chirac (*Center-Right*)	19.88%	82.21%
Jean-Marie Le Pen (*Far Right*)	16.86%	17.79%
Lionel Jospin (*Center-Left*)	16.18%	
François Bayrou (*Center*)	6.84%	
Arlette Laguiller (*Far Left*)	5.72%	
Jean-Pierre Chevènement (*Left*)	5.33%	
Noël Mamère (*Left*)	5.25%	
Olivier Besancenot (*Far Left*)	4.25%	
Jean Saint-Josse (*Right*)	4.23%	
Alain Madelin (*Center*)	3.91%	
Robert Hue (*Far Left*)	3.37%	
Bruno Mégret (*Right*)	2.34%	
Christiane Taubira (*Center Left*)	2.32%	
Corinne Lepage (*Center Left*)	1.88%	
Christine Boutin (*Center Right*)	1.19%	
Daniel Gluckstein (*Far Left*)	0.47%	

Table 7

6.8 Other Multi-Round Voting Systems

Standard Plurality Runoff Rule calls for a runoff unless a candidate wins a majority of the vote on the first round. Other Plurality Runoff rules set a threshold lower than 50% to trigger a runoff; for example, a 40% threshold is sometimes used to reduce the frequency of runoffs. Other possible thresholds pertain to vote relationships among the top three candidates. Under the *Double Complement Runoff Rule*, a runoff takes place only if the number (or percent) of votes by which the plurality winner falls short of a majority exceeds the number (or percent) of votes by which the plurality winner leads the runner up.

Most runoff systems eliminate candidates on the basis of who has the fewest first preferences among the surviving candidates, but other elimination rules can be used. The *Coombs Rule* eliminates the candidate with the most last preferences. Another possibility is to eliminate candidates with the lowest Borda score.[29]

Other multi-round voting systems do not ask voters to rank candidates and, rather than automatically eliminating candidates between rounds, allow voters to change their votes and/or candidates to withdraw (or even enter) between rounds. A *two-round majority-plurality system* is one example: the majority winner, if one exists, is elected in the first round; otherwise there is a runoff in which the plurality winner is elected — but it is expected that some or most trailing candidates will withdraw between rounds.[30] An *open-ended majority system* is another: balloting continues until one candidate is supported by a majority of votes cast.[31] We call these 'systems' rather than 'rules', since they do not determine the winner in an automatic fashion from a given ballot profile.

6.9 Approval Voting

We conclude this section by discussing one other voting system that does not select a definite winner from a given ballot profile. While it has never been used in any public elections, it has been widely discussed by social choice scholars and has been adopted by several scientific societies to elect officers.

Under *Approval Voting*, voters do not rank candidates but can vote for ('approve of') any number of candidates, and the candidate with the most such 'approval votes' wins.[32] Thus, in the three candidate cases a voter could vote for just one candidates (as under simple plurality) or for two. (It should be clear that voting for all three is effectively equivalent to abstaining.)

While approval voting has some advantages, it can be highly indeterminate. For example, given Profile 11 sincere approval voting can select Labour (if each voter votes for his most preferred candidate only), Conservative (if only voters in the 20-voter bloc cast two approval votes), or Liberal (if only voters in the 34-voter bloc cast two approval votes or if all voters cast two approval votes). Perhaps the most plausible scenario is that most Liberal voters would be more or less indifferent between the two other candidates (Labour being too far to the left and Conservative too far to the

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29 Under both of these rules, it is essential that voters rank all candidates, which in practice many voters may have difficulty doing in a meaningful way. (Voters' lowest preferences are less likely to come into play under AV.)

30 Such a system is used in French parliamentary elections but with the stipulation that candidates who fail to receive some minimum threshold of the vote (at present 12.5%) in the first round are automatically eliminated from the second.

31 When U.S. party presidential nominating conventions actually selected a party's nominee for president, they used such an open-ended system.

32 In effect, sincere voters could submit a standard ballot ranking but with an additional mark separating their 'approved' from 'non-approved' candidates.

right) and therefore would not vote for a second candidate, while some or many Labour and Conservative voters would also vote for the Liberal. But whether enough would do so to elect the Liberal is not clear.

Bibliographical Notes and Further Readings

There are many surveys of single-winner voting rules, including Straffin (1980), Nurmi (1983), Riker (1982), Brams and Fishburn (2002), and Hodge and Klima (2005). On the rivalry between Borda and Condorcet, see McLean and Urken (1995). Notable advocates of Borda rule include Dummett (1984), Saari (1995), and Emerson (2013). On runoff variants, see Grofman (2008); the Double-Complement Rule was proposed by Shugart and Taagepera (1994). Brams and Fishburn (1978 and 1983) are the most prominent analysts and advocates of Approval Voting.

7. Properties of Voting Rules

We now identify some conditions that we may want voting rules to obey and determine whether the rules that we have identified in fact do so.[33] It is evident that they all obey May's conditions of Anonymity and Neutrality.

7.1 Majoritarianism and Condorcet Consistency

All the rules we have considered are at least weakly majoritarian — that is, they are equivalent to Simple Majority Rule in the two-alternative case. However, Profile 13 shows that Borda Rule may fail to be simply majoritarian, as Conservative is the majority winner but Labour is the Borda winner.
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Profile 13

# of voters =>	46	54	Borda Ranking
1st pref.	*Labour*	*Conservative*	*Labour* (146)
2nd pref.	*Liberal*	*Labour*	*Conservative* (108)
3rd pref.	*Conservative*	*Liberal*	*Liberal* (46)

While Profile 13 demonstrates that members of the majority coalition of 54 voters cannot guarantee a Conservative victory simply by ranking him first, the question remains whether it can guarantee a Conservative victory in some other fashion. Note that if the 54 voters strategically move Labour to the bottom of their orderings, Labour's Borda score falls to 92, so Conservative's score of 108 is now enough to make Conservative the Borda winner. But this does not guarantee a Conservative victory, since the other 46 voters can strategically move Liberal to the top of their orderings and thereby give Liberal (whom they prefer to Conservative) a winning Borda score of 146. In general, the strongest strategy is for the majority coalition to rank Conservative first and to split their second preferences equally between Labour and Liberal, thereby minimizing their contribution to the Borda score of either candidate at 27. But the minority of 46 still give Labour an additional 92

[33] Module 12 examines many of these and other single-winner voting rules and their propensity to be subject to various social choice 'paradoxes' in greater detail.

Borda points for a total of 119 and thereby deny victory to Conservative.[34] Thus, the Borda Rule is not strictly majoritarian.

In contrast, Approval Voting is strictly majoritarian, as voters in any majority coalition can guarantee that a candidate is elected by voting for that candidate and no others. On the other hand, a majority winner *A* may fail to be elected under Approval Voting, since some voters in the majority who most prefer *A* may also vote for another candidate *B*, and these approval votes for *B* together with those cast for *B* by other voters may result in the election of *B*. The other voting rules we have considered always elect a majority winner.

We have seen that, given three or more candidates, the Condorcet winner may have the fewest first preferences and, given four of more candidates, no first preferences at all. This implies that a voting rule that takes account of first preferences only cannot be Condorcet consistent. In addition to Plurality Rule, this applies to rules such as Plurality Runoff and the Alternative Vote under which candidates are eliminated on the basis of first preferences (with respect to surviving candidates in the case of AV). Profiles 11 and 13 demonstrate that the Borda Rule is not Condorcet consistent, and the indeterminacy of Approval Voting means that it is not Condorcet consistent either.

7.2 Strategyproofness

To this point we have mostly assumed that voters vote sincerely — put otherwise, that ballot profiles are identical to preference profiles. But every voting rule with three or more candidates may give voters incentives to vote otherwise than sincerely, i.e., fails to be *strategyproof*. We have already seen that various strategic moves and countermoves are available under Borda Rule when we considered whether it is strictly majoritarian.

Consider Profile 11 again. As we saw, Labour wins under Plurality Rule if voters are sincere. But it is also true that a majority of 54 voters prefer both other candidates to Labour, who is thereby the Condorcet loser. If they all vote for the *same* other candidate, that candidate wins — an outcome they all prefer to a Labour victory. But doing this requires some members of this majority of 54 to vote 'insincerely' for their second preferences. Thus Plurality Rule (as well as other voting rules) can encourage what the British (whose general elections feature three or more candidates in many constituencies) call 'tactical voting' and what most political scientists call *strategic voting*.

Of course, the further problem is *how* the majority of 54 voters coordinate their votes — that is, will they vote for Liberal or for Conservative? While all 54 voters prefer to see Labour defeated, they disagree as to *how* to defeat him. The general notion is that supporters of candidates third or lower in the prospective (sincere) plurality ranking (as estimated from polls or otherwise) conclude that a sincere vote is 'wasted' and that they should vote 'tactically' for their more preferred of the

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[34] It can be checked by simple algebra that, given 100 voters and three candidates, it takes a majority of 58 voters or larger to guarantee victory for its preferred candidate under the Borda Rule. (This assumes that all voters must rank all candidates and thereby avoids the question of how 'truncated' ballots should be counted under Borda rule; on this issue, see Emerson, 2013.)

two leading candidates. In British constituency elections, the Liberal candidate typically trails the Conservative and Labour candidates in first preference support (as in Profile 11).[35]

Under Plurality Runoff (or AV), the 46 voters who most prefer Labour would do better by ranking Liberal first, as this brings about a Liberal victory without a runoff, which outcome they prefer to the Conservative victory that would otherwise result. Moreover, no countermove is available to Conservative supporters.

Given Profile 11 and Borda Rule, no voters can change their ballot rankings in a way that improves the outcome for them; while either the Labour-preferring bloc of 46 or the Conservative preferring bloc of 34 can deny Liberal victory by pushing Liberal to the bottom of their rankings, the result is to elect their least preferred candidate; and if they both did this simultaneously, Labour is elected, an outcome that Conservative supporters would very much regret.

However, Borda Rule is in general extremely susceptible to strategic voting. Given Profile 12, if the bloc of 20 voters switches the ranking of Conservative and Liberal *and* the bloc of 34 switches the ranking Liberal and Labour, the result is that Conservative wins with a Borda score of 108 (vs. 100 for Liberal and 92 for Labour), an outcome all 54 such voters prefer to victory by the sincere Borda winner Labour. And we have seen that Profile 13 provides a glaring opportunity for strategic voting under Borda Rule. If voting is sincere, Labour is the Borda winner, but the 54 Conservative-preferring voters can elect Conservative if they shove Labour down to third place on their ballots. However, the 46 Labour-preferring voters can counteract this by moving Liberal to the top of their ballots — the resulting Liberal victory being preferable to the 46 voters to a Conservative victory. Note that if strategic manipulation stops at this point (though it need not), Liberal is elected even though *everyone* prefers Labor to Liberal.

And we can devise an even more perverse example of strategic voting under Borda voting. Suppose there are three candidates: a more or less reasonable Labour candidate, a more or less reasonable Conservative candidate, and the candidate of the Monster Raving Looney Party, who has a couple of deranged supporters.

Profile 14

# of voters =>	50	48	1	1
first pref.	*Labour*	*Cons.*	*Looney*	*Looney*
second pref.	*Cons.*	*Labour*	*Labour*	*Cons.*
third pref.	*Looney*	*Looney*	*Cons.*	*Labour*

35 Moreover, in general elections Liberal candidates are further weakened by the fact, even if a Liberal candidate is doing well in a particular constituency, national polls typically show Liberals trailing substantially behind both other parties in the national vote. Notice that this weakness occurs even if Liberal is the Condorcet winner in a particular constituency and possibly nationally (as in Profile 11), reflecting the fact that polls (almost always) ask only about first preferences and Liberal's strength typically lies in second preferences (as in Profile 11). However, if Liberal supporters find Labour and Conservative to be equally objectionable, they have no incentive to vote tactically and, if pre-election polls show something close to a tie for second place (or a three-way tie), tactical voting becomes far more conjectural.

If everyone votes sincerely, Labour wins (the same outcome as under Plurality Rule) with 149 Borda points versus 147 for Conservative and 4 for Looney. Anticipating defeat, Conservative voters notice a devastating feature of Borda rule — it can pay voters to engage in 'turkey raising', i.e., strategically raising an outrageous third candidate (a 'turkey') in their ballot rankings, so as to push the rival 'serious' candidate down in their rankings and thereby increase the Borda point spread between the two 'serious' candidates generated by their ballots. Suppose the Conservatives strategically modify all their ballots in this fashion. While Conservative still has a Borda score of 147, Labour's has fallen to 101 (and Looney's has increases to 52), producing a clear Conservative victory. But suppose that before the actual balloting takes place, Labour supporters also notice this feature of Borda Rule and, fearful that Conservatives will engage in turkey raising, determine that they must engage in turkey raising of their own in order to counteract the anticipated Conservative stratagem. The Borda scores are now 101 for Labour, 97 for Conservative, and 102 for Looney.

7.3 Spoilerproofness

Consider an individual who is authorized to dictate choice of a candidate unilaterally and, when given a choice between Conservative and Labour only, chooses Conservative. We would think this individual mighty peculiar if he switched his choice to Labour in the event Liberal is added as a third option. But a group of sincere voters using Plurality Rule may do exactly this, as can be verified by checking Profiles 11 and 12. To take a real world example, Albert Gore would have almost certainly won Florida's electoral votes (and the U.S. Presidency) in 2000 if Ralph Nader had not been on the ballot in Florida. Strategic voting may mitigate this problem, however. Undoubtedly many left-of-center voters who sincerely preferred Nader to Gore nevertheless voted for Gore in order to (try to) preclude a victory by the even more disliked George W. Bush, and if a few more had done so the tactical voting would have succeeded.

But such 'spoiler effects' are not peculiar to Plurality Rule. Borda Rule is similarly vulnerable to spoiler effects, as can be verified by checking Profile 11. Plurality Runoff Rule, and more particularly AV, are often advocated on the grounds that they preclude spoiler effects. It is true that these voting rules are superior to Plurality Rule in that a third candidate with little first-preference support (such as Nader) cannot act as a spoiler in what is virtually a straight fight between two major candidates, because the runoff will become precisely that straight fight. However, Plurality Runoff Rule and AV do not eliminate the spoiler problem if first preferences are more equally distributed, as is illustrated by Profile 11. Under AV, Liberal wins a straight fight with Conservative but loses to Conservative if Labour enters the field, making Labour a spoiler against Liberal.

Variants of Condorcet voting are immune to the spoiler problem if — but only if — there is a clear Condorcet ranking based on the pairwise majority preference relation (which takes no account of the presence or absence of other alternatives). However, any variant of Condorcet voting must make provision for identifying a winning candidate in the face Condorcet cycles and, in doing so, the spoiler problem arises again.

7.4 Treatment of Clone Candidates

The spoiler problem is likely to be especially prominent when some candidates are (more or less) 'clones' of each other. Informally and substantively, two candidates are 'clones' when they occupy essentially the same position on the ideological spectrum or favor the same positions on almost all issues, and are not particularly distinguishable in terms of their personal qualities, party affiliation, etc., so that voters who like (or dislike) one like (or dislike) the other almost equally. Formally, two candidates are defined as *clones* when they are adjacent in the preference orderings of all voters.

Consider Profile 15, in which a right-of-center minority is united behind a single candidate *R* but a left-of-center majority is split between the two clone candidates *L1* and *L2*.

Profile 15

Left	*Right*	*Left*	*Right*
35%	25%	25%	15%
L1	*L2*	*R*	*R*
L2	*L1*	*L1*	*L2*
R	*R*	*L2*	*L1*

Plurality Rule is notorious for penalizing clone candidates. In this case, the *R* candidate would win due to the split in the left, even though *R* is the Condorcet loser who would be beaten by either *L1* or *L2* in a straight fight.[36] (Indeed, *R* is ranked last by a majority of voters.) The question arises of whether there are other voting rules that can reduce, eliminate, or even reverse the self-defeating effect of running clone candidates. We may note that, given the profile above, Plurality Runoff Rule solves the clone problem to the advantage of the left-of-center majority. But if there are four or more candidates, Plurality Runoff does not treat clones so well. Advocates of Approval Voting cite among its advantages the fact that it does not punish clones. In the profile above, presumably (almost all) right-of-center voters would vote only for *R* while (almost all) left-of-center voters would vote for both *L1* and *L2*, so one of the latter would be elected.[37]

A variation of one type of (open) party-list Proportional Representation (as discussed in Modules 7 and 11) applied to single-member districts presents another voting rule that does not penalize candidates who are clones by virtue of sharing the same party affiliation. Each voter votes for a single candidate, as under Simple Plurality, but this vote counts in two ways: first, as a *party vote* (here for either Left or Right) to determine which party wins the election and, second, as a *candidate vote* to determine which candidate of the winning party is elected. Given Profile 15, *L* would win the party election and *L1* would be the elected candidate.

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36 It is precisely the expectation of such outcomes under Plurality Rule voting that leads to political party formation and party discipline — that is to say, in this case left-of-center voters have a huge incentive to hold a prior party *primary election* (or *nominating convention* or *candidate adoption meeting*) to choose between *L1* and *L2* and then send just one of the two clones forward against *R* in a straight fight general election. This logic is further pursued in Module 7.

37 By not penalizing clones, Approval Voting does not encourage party formation or party unity. For this reason, many American political scientists are more inclined to support Approval Voting for party primaries and non-partisan elections than for general (partisan) elections.

Perhaps surprisingly, the Borda Rule actually *rewards* the running of clones. Suppose that there are two candidates and the Right party is again in the minority, as shown in Profile 16

Profile 16

60 voters	40 voters
L	*R1*
R1	*L*

Profile 17

60 voters	40 voters
L	*R1*
R1	*R2*
R2	*L*

With just two candidates, Borda Rule is identical to Plurality Rule, so *R1* loses. But, if Borda Rule is in use, the Right party can reverse the outcome by nominating an additional clone candidate *R2* whom everyone sees as identical to *R1* with respect to issues and ideology but slightly inferior with respect to (let's say) personal qualities, as shown in Profile 17.

Now *R1* wins with a Borda score of 140 to 120 for *L* and 40 for *R2*. Of course, the Left party can counteract this by strategically ranking *R2* above *R1*, thereby reducing *R1*'s score to 80 points and raising *R2's* to 100, allowing *L* to win with an unchanged score of 120 points. Alternatively, it can counteract the Right's stratagem by running its own clone candidate. Though it has strong advocates, Borda Rule is evidently highly susceptible to strategic maneuvers of this sort — which, moreover, have the effect of expanding the candidate field rather than winnowing it down in the manner of Plurality Rule.

7.5 Monotonicity

Runoff voting rules (including AV) not only do not wholly preclude spoiler effects but have an additional and distinctive flaw. We wouldn't expect a voting rule to respond *negatively* when a candidate's position in a preference profile becomes unambiguously more favorable — increased support should never hurt a candidate. This condition is commonly called *monotonicity* and it extends May's non-negative responsiveness condition to the multi-alternative case. Strangely and unexpectedly, Plurality Runoff Rule and AV (and other elimination rules) fail to obey monotonicity.

Suppose that we have three candidates *A*, *B*, and *C* (so Plurality Runoff and AV are equivalent), among whom first preferences are fairly equally divided. Suppose that *A* and *B* go into the runoff, which is therefore decided by the second preferences of the voters who most prefer *C*. Suppose that enough of these second preferences are for *A* that *A* wins the runoff. Now suppose that the preference profile is revised in a way that makes 'public opinion' even more favorable to *A* (without changing anyone's preferences between *B* and *C*). In particular, suppose that some voters who previously ranked *B* first now move *A* up into this position (but *A* still is not a majority winner), while no other preferences change. The result may be that the number of first preferences for *B* falls below the number for *C*, with the result that *A* and *C* are now paired in the runoff, which is decided by the second preferences of the (remaining) voters who most prefer *B*. And it may be that enough of these second preferences are for *C* that *C* rather than *A* wins the runoff. Thus, added support has cost *A* victory. The two versions of Preference Profile 18 provide a specific example in which the set of 10 voters shifts their ballots in *A*'s favor, with the result that the initial winner *A* is defeated.

Original Profile 18

# of voters	35	10	25	30
first pref.	*A*	*B*	*B*	*C*
second pref.	*B*	*A*	*C*	*A*
third pref.	*C*	*C*	*A*	*B*

Revised Profile 18

# of voters	35	10	25	30
first pref.	*A*	*A*	*B*	*C*
second pref.	*B*	*B*	*C*	*A*
third pref.	*C*	*C*	*A*	*B*

While this example pertains to both Plurality Runoff and AV, with four or more candidates the former may be vulnerable to monotonicity problems when the latter is not. Referring to the 2002 French presidential election results presented in Table 7, consider fairly right-wing voters who were on the borderline between most preferring the respectable center-right candidate Chirac and the unrespectable far right candidate Le Pen, and suppose that some of these voters who actually voted for Le Pen had switched to Chirac, while no other voters change their preferences. Thus 'public opinion' becomes clearly more favorable to Chirac. Yet this shift might cause him to lose the election. If Le Pen's first-preference support falls enough to put his vote below Jospin's, Jospin rather than Le Pen would have gone into the runoff. In this runoff, the left-of-center vote probably would have largely consolidated behind Jospin and (depending on how centrists would vote in the runoff) Chirac might have lost as a result of his increased support. On the other hand and as noted earlier, under full AV the final 'runoff' would almost surely have been between Chirac and Jospin in any case.

Profile 19 illustrates a related quirk of Plurality Runoff Rule and AV called the *no-show paradox*.

Profile 19

# of voters =>	5	6	4	[2]
first pref.	*B*	*C*	*A*	[*A*]
second pref.	*C*	*B*	*B*	[*B*]
third pref.	*A*	*A*	*C*	[*C*]

Suppose that the two individuals with the bracketed preference orderings fail to vote. Thus, the election outcome is determined by the remaining 15 voters. Candidates *B* and *C* are paired in a runoff, which *B* wins. Because the winning candidate was only their second preference and thinking that their first preference *A* might have won if they had not failed to vote, the two individuals who failed to vote greatly regret their failure to get to the polls. But it can be checked that, if they had gotten to the polls and voted according to their preferences, the outcome would have been worse for them, not better, as candidates *A* and *C* would have been paired in a runoff which *C* would have won.

AV and similar rules also have the property that they may not respond in the expected fashion to general shifts in voter opinion. Suppose that voters and candidates are both arrayed along an ideological spectrum, and that voters preference single-peaked. We would expect (correctly) that, if voter opinion is sufficiently skewed to the left, the most left-wing candidate will win under any voting rule, and likewise if public opinion is sufficiently skewed to the right. Beyond this, we would expect that a voting rule would respond monotonically to shifts in public opinion — that is to say, if public opinion were to shift incrementally from left to right (or right to left) while candidate positions remain fixed, the ideological position of the winning candidate would likewise shift from left to right (or right to left), though possibly skipping over one or more intermediate candidates. While Plurality Rule is notorious for 'squeezing' out centrist candidates in multi-candidate elections, it responds monotonically to shifts in public opinion. But Plurality Runoff and AV do not.

Ordering	*L* *C* *R*	*C* *L* *R*	*C* *R* *L*	*R* *C* *L*	*Runoff*	*Winner*
Profile 1	13	6	1	5	—	L
Profile 2	12	5	2	6	L vs. C	C
Profile 3	10	4	3	8	L vs. R	L
Profile 4	8	3	4	10	L vs. R	R
Profile 5	6	2	5	12	C vs. R	C
Profile 6	5	1	6	13	—	R

Table 8

Consider the six preference profiles over candidates *L* (leftist), *C* (centrist), and *R* (rightist) for 25 voters shown in Table 8. Public opinion is most left-wing in Profile 1 and becomes steadily more right-wing moving downward. While as expected candidate *L* wins under Profile 1, candidate *R* wins under Profile 6, and candidate *C* wins under some intermediate profiles, the rightward movement from Profile 2 to 3 produces a leftward movement in the winner and likewise for Profiles 4 and 5.

Bibliographical Notes and Further Readings

Heckelman (2015) provides an introductory examination of the properties of single-winner voting rules, while Nurmi (1987 and 1999) and Felsenthal and Nurmi (2017 and 2018) provide more detailed examinations. The claim that no (deterministic) voting rule can be strategyproof given three or more alternatives is usually based on Gibbard (1973) and Satterthwaite (1975), but also see Duggan and Schwartz (2000). The similar claim pertaining to spoilerproofness has no similar basis in a formal theorem but rests essentially on arguments that go back to Borda and Condorcet and is loosely implied by Arrow's (1963) theorem (see Module 5). The example of 'turkey raising' under Borda Rule is drawn directly from Monroe (2001). The concept of clone candidates was introduced by Tideman (1987). Miller (2017) sets out conditions under which Plurality Runoff Rule (or AV) is subject to monotonicity problems, along with estimates of the frequency with which they hold; also see Ornstein and Norman (2014). Duddy (2017) provides a geometric explanation of how such paradoxes occur and proposes a runoff rule that is not subject to the problem. The example shown in Table 8 was suggested by Yee (2006).

8. Sequential Pairwise Majority Voting

The voting rules discussed to this point may conveniently be used in elections in which voters cannot be expected to cast repeated ballots. But in smaller voting bodies such as committees, public meetings, legislative assemblies, and parliaments, it is feasible and standard practice to take a sequence of pairwise votes such that the nature of later choices depends on the results of earlier votes. Each vote, being pairwise, is tabulated on the basis of Simple Majority Rule; individual votes thus avoid the problems that afflict the multi-alternative voting rules discussed in the previous section. But complexities arise in structuring the sequence of votes that leads to the final voting outcome and, in this respect, similar problems reappear. Given that they are based on pairwise majority rule, sequential voting rules generally comply with the spirit of Condorcet's pairwise approach to voting, and they are typically (but not always) Condorcet consistent.

8.1 Ordinary Committee Procedure

The formal analysis of sequential pairwise majority voting originated with the work of Duncan Black (1948, 1958). Black (1948: 25) spoke of voting taking place in this manner, which he referred to as 'ordinary committee procedure':

> In practice, voting would be so conducted that, after discussion, one motion would be made and, after further discussion, another motion (an 'amendment,' that is) might be moved. If so, the original motion and amendment would be placed against each other in a vote. One of the two motions having been disposed of, leaving a single motion in the field, a further amendment to it might be moved; then a further vote would be taken between the survivor of the first vote and the new motion; and so on.

Thus some process defines the set of alternatives and the order in which they are paired for votes. Let's label the alternatives $x_1, x_2, \ldots, x_m$ according to the order in which they enter the voting. Under *ordinary committee procedure*, the first vote is between x_1, and x_2, the second between the winner of the first vote and x_3, and so on; the winner of the final or $(m-1)^{\text{th}}$ vote is the winning outcome. Since alternatives are explicitly paired for votes, 'sincere' voters simply vote for the alternative that is higher in their preference orderings and, given the majority preference tournament and the order of voting, we can readily trace out which alternative wins each vote and determine the final voting outcome.

It should be clear that ordinary committee procedure is Condorcet consistent — that is, the Condorcet winner, if one exists, is the final outcome regardless of the order in which alternatives enter the voting. The Condorcet winner must enter the voting at some point and it defeats whatever alternative it is first paired with and every alternative it is paired with thereafter. A generalization of this logic implies that, if there is no Condorcet winner, the final outcome is some alternative in the top cycle set. The first top cycle alternative to enter the voting defeats the winner of the preceding vote and can subsequently be defeated only by another top cycle alternative.[38] Generalizing further,

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[38] Or if either x_1 or x_2 (or both) are top cycle alternatives, one of them wins the first vote, and thereafter the same logic applies.

it is clear that the existence of non-top cycle alternatives and their place in the voting order have no effect on the final outcome — all that matters is the order of voting among top cycle alternatives, which determines which of them wins.

The conclusion that *only* an alternative in the top cycle set can be the final outcome under ordinary committee procedure leaves open the question of whether *every* such alternative can be the final outcome. In fact, this is true also. As noted in 4.5, every top cycle alternative x is the first element in some complete path in the majority preference tournament; if we let the voting order be the reverse of this complete path, x enters the voting last and defeats the alternative with which it is paired. Thus, for every top cycle alternative x, there is some voting order that places x last that makes x the winning outcome. Note that this conclusion implies that alternatives outside the Pareto set may become the voting outcome under ordinary committee procedure. For example, in Profile 7 alternative z is unanimously preferred to v; nevertheless, v belongs to the top cycle and wins if the voting order z, y, x, v.

In general, an alternative is advantaged by entering the voting late, since it then has to defeat fewer other alternatives, which implies that ordinary committee procedure, unlike the voting rules discussed earlier, violates May's neutrality condition. But this claim about voting order can be made more precise in several ways. First, putting x last among top cycle alternatives in the voting order is neither necessary nor sufficient for x to be the voting outcome; for example, given Profile 8 and the voting order v, z, x, y, alternative x is the voting outcome. Second (to repeat a point made earlier), all that matters is the order in which top cycle alternatives enter the voting. Third, neither of the first two top cycle alternatives that enter the voting can be the final winning outcome, since one is defeated by the other and the latter must be defeated by third top cycle alternative that enters the voting later. Finally, if alternative x is the voting outcome under a given voting order, x is also the winning outcome under any other voting order that is identical to the first with respect to the alternatives that precede x.

Because only $m-1$ votes are taken among the $m(m-1)/2$ pairs of alternatives, ordinary committee procedure 'reveals' majority preference between some pairs of alternatives but not the whole majority preference tournament. More precisely, the voting reveals only that, if x_k is the winning alternative, x_k beats one alternative that precedes it and all that follow it in the voting order and that every other alternative is beaten by one other alternative. This means that the existence of a Condorcet cycle cannot be revealed and that, even if there is a Condorcet winner, this fact is revealed only if it enters the very first vote.

8.2 Variants of Ordinary Committee Procedure

Ordinary committee procedure is 'continuous' in that the winner of a given pairwise vote always appears in the subsequent pairwise vote. 'Discontinuous' variants of the procedure allow for a vote on one pair of alternatives, then a vote on a quite different pair, with the winners of these two votes being paired at some subsequent vote. Given m alternatives, this variant still results in $m-1$ rounds of voting but they are differently sequenced.

What difference does such 'discontinuity' make for voting outcomes? It is clear that discontinuous variants are also Condorcet consistent and, more generally, guarantee that some top cycle alternative is the voting outcome. But it is no longer the case that every top cycle alternative can win. Consider again the majority preference tournament for Profile 8 in Figure 1 and any discontinuous variant of ordinary committee procedure. There are only three possibilities: *x* vs. *y* and *z* vs. *v* with the winners paired at the final vote; *x* vs. *z* and *y* vs. *v*; and *x* vs. *v* and *y* vs. *z*. Thus there can be at most three (not four) possible voting outcomes. And it can be checked that *x* wins under both of the first two variants, while *y* wins under the third. In this case the possible winners are restricted to the two alternatives in the Copeland set because, in order to be the final outcome, an alternative must beat two other alternatives and only *x* and *y* do this (and *x* is further advantaged because it beats the other Copeland winner).

Bibliographical Notes and Further Readings

As indicated at the outset of this section, the seminal work is Black (1948, 1958). The scope of Miller (1995) is defined '*committee voting* — that is, voting in a parliamentary context, in which collective choice proceeds through a sequence of *binary*, (e,g., yes/no) votes', where 'binary' is equivalent to 'pairwise'. As such, its scope corresponds to Sections 8-10 of this module.

9. Parliamentary Voting

Given its pairwise nature, ordinary committee procedure bears considerable resemblance to voting of the parliamentary type but does not capture its essential nature. Parliamentary procedures allow legislative proposals to be moved on a particular topic and then specify some *agenda* of 'questions' that are put to yes vs. no votes in a definite order and ultimately lead to the selection of some alternative. Typically the questions are voted on in an order that is the reverse of the order the motions were proposed. Moreover, the relationship between proposals and alternatives is not entirely straightforward.

9.1 Parliamentary Agendas

The set of alternatives is generated by *proposals* made by members in the form of motions. These proposals may be 'bills', 'amendments', 'substitutes', etc., according to the order in which they are proposed and relevant parliamentary usage. These designations influence the structure of the agenda and, in particular, the order in which questions are voted on. However, there are typically more alternatives than proposals. In particular, the alternative of 'doing nothing', even though not explicitly proposed, is almost always available as an outcome; this inaction has the effect of maintaining the *status quo* or effecting a reversion to some pre-established alternative (e.g., some level of taxation and/or expenditure). We refer to this as the *status quo alternative* and designate it *q* (as in Profile 10). Thus, even if just a single bill *b* is proposed, the voting body faces a choice, i.e., whether to accept or reject the bill, so two alternatives *b* and *q* are on the agenda. As we shall see, there are other ways in which the number of alternatives may exceed the number of proposals.

Farquharson (1969, p. 9) provided the first formal definition of parliamentary voting procedures in this manner:

> We shall define a *voting procedure*, initially, in terms of the set of [alternatives or possible] outcomes. Suppose this set is divided into two subsets, each subset into two further subsets, and so on until single outcomes are reached. Such a sequence of divisions may be depicted as

a tree, the 'outcome tree'. Each of its forks corresponds to a division [i.e., a pairwise vote] and the end of each of its branches corresponds to an outcome.

Intuitively a 'tree' is a branching structure with a single starting point and multiple end points. Recall from Box 10 that a *tree* is special type of directed graph and that a binary tree has exactly two nodes following each non-terminal node. Figure 6 provides an example of a somewhat complex (and non-uniform) binary tree.[39] Since it prescribes a sequence of yes/no votes, any parliamentary voting procedure can be represented by a binary tree in which (i) every terminal node is associated with an 'outcome', i.e., a winning alternative, and (ii) the alternatives associated with a pair of terminal nodes that follow the same node are distinct. Each non-terminal node is associated with the outcomes reachable from it and therefore the initial node is associated with the whole set of possible voting outcomes. Each non-terminal node represents a possible yes/no vote, which is in effect a choice between the alternatives associated with the two following nodes. Figure 6 shows an arbitrary agenda tree with eight alternatives.[40] It does not represent any plausible voting procedure but it is consistent with the discussion of parliamentary agendas to this point.

We say that an alternative associated with a given node is *unchallenged* at that vote if it belongs to both following subsets, since it remains as possible outcomes regardless of the result of the vote. In contrast, an alternative that belongs to one set but not the other is *challenged* at that vote, because it may be eliminated as a possible outcome depending on the result of the vote. Thus in Figure 6, x_1, x_2, x_7, and x_8 are challenged at the first vote, while x_3, x_4, x_5, and x_6 are unchallenged. At nodes 2.1 and 3.1, every alternative that remains as a possible outcome is challenged, while node 5.1 poses a pairwise choice between just two challenged alternatives x_3 and x_5.

Two broad types of parliamentary voting procedures are in common use, which Farquharson dubbed *amendment procedure* and *successive procedure*. The former is used primarily in English-speaking countries; variants of the latter are common in much of continental Europe and Latin America. Under successive procedure each proposal is a complete bill, and the rival bills are voted on in some order; the status quo prevails if every bill is rejected. In contrast, under amendment procedure, only the first proposal is a complete bill, but various types of amendments to this bill may be proposed, so the array of amendments, in conjunction with the original bill and the status quo, generates the set of alternatives. The amendments are voted on in some order (typically the reverse of the order in which they were proposed) and at the final vote the bill (as it may have been amended) is voted up or down; if it is rejected, the status quo prevails.

9.1 Simple Amendment Agendas

The U.S. Congress uses agendas of the amendment type. As an example, let us consider (a simplified version of) the Powell amendment discussed in 5.6. Under the status quo, the federal government provides no aid to states for education. A Democratic bill, opposed by most Republicans, would authorize such aid. Representative Powell's amendment to the bill would deny such aid to

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[39] For the moment, the reader should ignore everything other than the nodes and arrows.

[40] For the moment, the reader should ignore the fact that in Figures 6 and 7(a) one alternative associated with each non-terminal node is underlined.

(Southern) states with racially segregated schools. These two proposals generate three alternatives: the original bill *b*, the bill with Powell amendment attached *b*+*a*, and the status quo *q*. Under amendment procedure, the first vote is on the question of adopting the amendment, and the second vote is on the question of adopting the bill (as amended or not). The voting tree for this example is shown in Figure 7(a); here and elsewhere we shall adopt the convention that a winning 'yes' vote leads to the left following node, while a winning 'no' vote leads to the right following node. Thus, initially all three alternatives are possible outcomes; as a result of the first vote, either *b*+*a* (if the amendment is adopted) or *b* (if the amendment is rejected) remain as possible outcomes along with *q*; as a result of the second vote, only one possible outcome remains, i.e., the winning alternative.

Farquharson (1969, pp. 17-19) also provided a definition of 'sincere voting' in the parliamentary context: a *sincere vote*r votes 'yes' at a given division if the left following node contains his most preferred challenged alterative and votes 'no' if the right following node contains his most preferred challenged alterative. Given sincere voting on the basis of the preferences shown in Profile 10, the amendment is adopted and then the bill as amended is rejected, so the status quo prevails (which is what actually happened in the 1950s).

Anticipating this outcome, supporters of federal aid on a non-segregated basis, hoping to attract enough Southern support to pass the bill, might have proposed a 'softening' amendment a' to the Powell amendment specifying that a state with segregated schools can receive federal aid if it promises to desegregate its schools within a specified number of years. In this case, the first vote, would be on the question of adopting the amendment to the (Powell) amendment, the second vote on the question of adopting the Powell amendment (as amended or not), and the final vote would be on the question of adopting the bill (as amended or not), giving the voting tree shown in Figure 7(b).

Generalizing this pattern leads to the definition of a *simple amendment agenda*, in which a bill is proposed, then an amendment to the bill, then an amendment to the amendment, then an amendment to the second amendment, and so on. We can make a number of generalizations about simple amendment agenda trees. First (in contrast to the agenda tree in Figure 6), they are *uniform* in that each path from the initial node to any final node is of the same length; put substantively, given *m* alternatives, exactly $m-1$ votes are taken (regardless of the results of earlier votes). Second (again in contrast to the tree in Figure 6), they are *complete* in that the size of the alternative set associated with each node is reduced by just one alternative each step down any branch of the tree; put substantively, the result of every vote is to eliminate one and only one alternative from the set of possible outcomes, so exactly two alternatives at each node are challenged; the effect is that every vote is in effect a pairwise choice between two (challenged) alternatives. Third, they are *continuous* in that, once an alternative is challenged at a vote, it remains challenged at subsequent votes until it is eliminated as a possible outcome or wins the final vote.

The upshot is that the structure of a simple amendment agenda is logically equivalent to that of ordinary committee procedure, but with this practical point of contrast. Under ordinary committee procedure, the voting agenda is 'forward moving', i.e., an initial proposal is put against the status quo; then another proposal is put against the (perhaps new) status quo, i.e., the winner of the previous vote; and so forth. In contrast, a simple amendment agenda is 'backwards built'. Proposals are made in a particular order: the status quo is 'on the agenda' at the outset, then a bill is proposed, then an amendment to the bill, then an amendment to the amendment, and so forth. No voting takes place until all proposals have been made, at which time alternatives are voted on in the reverse order in which they were proposed: e.g., first, on the question of the amendment to the amendment, then on the question of the amendment (as amended or not), and finally the question of the bill (as amended or not). At each vote, the yes/no choice is effectively a choice between two (challenged) alternatives, whereas under ordinary committee procedure voters are choosing between explicitly paired alternatives.

Thus, the previous conclusions pertaining to ordinary committee procedure apply also to simple amendment agendas: they are Condorcet consistent; in the absence of a Condorcet winner, only alternatives in the top cycle can win; which such alternative wins depends on the order in which they are voted on (and thus on the order in which they are proposed); and every top cycle alternative can win given an appropriate voting order. Note, however, that parliamentary circumstances preclude certain voting orders. Given the Powell Amendment example and Profile 10, all three alternatives belong to the top cycle, and the particular voting order leads to the status quo as the outcome. But the only other feasible parliamentary circumstance would be to include the substance of Powell amendment (prohibiting federal aid to segregated schools) in the original bill, in which case an amendment might be proposed to remove the prohibition; but (if we continue to assume sincere voting) such an amendment would fail, and then the unamended bill would fail, again preserving the status quo. Under ordinary committee procedure, the voting order that would make the unamended bill the winning outcome would require q and b+a to be paired at the first vote, and the order that make the amended bill the outcome would require that q and b be paired at the first vote, but under amendment procedure neither voting order is feasible as some version of the bill is always paired with the status quo at the final vote.

9.2 Two-Stage Amendment Agendas

After a bill b has been proposed (perhaps along with an amendment a to the bill), it may be in order to introduce a 'substitute bill' s — that is, another complete bill — and perhaps then an amendment a' to the substitute. This parliamentary situation introduces a 'discontinuity' into the agenda similar to that discussed with reference to ordinary committee procedure. The first vote would be on the amendment to the original bill; the second vote would be on the amendment to the substitute bill; the third vote would be on the question of the substitute bill (as amended or not) as a replacement for the original bill (as amended or not); and the final vote would be on acceptance of the pending bill (original or substitute, as amended or not). The agenda tree is shown in Figure 8. Note that while the agenda is complete (four votes on five alternatives), it is discontinuous in that b+a and b are challenged at the first vote but the winner is not challenged at the next vote. This is the simplest example of what may be called a *two-stage amendment agenda*, under which an original bill and amendments to it are considered in the manner of a simple amendment agenda, a substitute bill and amendments are likewise considered, a choice is then made between the two versions of the bill, and a final choice is made between the surviving bill and the status quo. More than one substitute bill might be proposed, generating a multistage amendment agenda, or a substitute amendments might be proposed, generating one two-stage agenda nested within another.

As we saw in connection with discontinuous versions of ordinary committee procedure, such agendas are Condorcet consistent and, in the absence of a Condorcet winner, allow only top cycle alternatives to win. But, in contrast to simple amendment agendas, a two-stage amendment agenda does not allow every top cycle alternative to win, since a winning alternative other than the status quo must have a sufficiently high Copeland score to be able to beat at least one alternative from each stage as well as the status quo.

9.3 Agendas with Compatible Amendments

The number of alternatives exceeds the number of proposals because the status quo is always implicitly 'on the agenda'. The number of alternatives further exceeds the number of proposals whenever two or more *compatible* amendments are proposed — that is, amendments that are literally not *alternatives* to one another, because they are not mutually exclusive. Suppose a bill b is first proposed and then two compatible amendments a_1 and a_2 are proposed. These three proposals generate an agenda with five alternatives: (1) adoption of the bill with both amendments ($b+a_1+a_2$), (2) adoption of the bill with the first amendment only ($b+a_1$), (3) adoption of the bill with the second amendment only ($b+a_2$), (4) adoption of the unamended bill (b), and (5) rejection of the bill (q). The agenda is shown in Figure 9.[41]

While this agenda is uniform and continuous, it is not complete (three votes with five alternatives) because, at the first vote, four alternatives (all but q) are challenged, not just two as under the agendas previously discussed. Since voters are not choosing between just two challenged alternatives, it may be less clear what constitutes 'sincere' voting.[42] However, applying Farquharson's definition, a sincere voter votes 'yes' at the first vote if his first preference among the challenged alternatives is either $b+a_1+a_2$ or $b+a_1$ and 'no' if his first preference is either $b+a_2$ or b. In particular, a voter whose first preference is $b+a_1$ would vote 'yes' on a_1 even if he prefers $b+a_2$ to $b+a_1+a_2$, so that he would vote against a_1 if a_2 had already been attached to the bill. Thus the outcome of the first vote depends on the distribution of first preferences over the challenged alternatives and not just on the majority preference tournament. This implies sincere voting on agendas with compatible amendments is not Condorcet consistent. However, if we assume that preferences over compatible amendments are *separable by amendments* (in the same way we described preferences as separable by issues in 4.3), thereby ruling out the kind of preferences just described, each compatible amendment is adopted if and only if it is majority- supported, and the overall process is Condorcet consistent and, in the absence of a Condorcet winner, leads to an outcome in the top cycle set. However, it is not true that any top cycle alternative can win given an appropriate voting order, as the outcome is always the bill plus any majority-supported amendments attached or the status quo, whereas additional alternatives may belong to the top cycle (as indicated by Profile 9).

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41 Standard parliamentary procedure would require that a_1 be voted up or down before amendment a_2 is formally proposed. Typically, however, it would be generally known at the time a_1 is voted on that a_2 will be proposed. In any event, we assume that the whole voting agenda is known before any voting takes place.

42 Note that a similar problem arises at many nodes in Figure 6.

9.4 Successive Agendas

As noted earlier, under successive agendas as used in most continental European and other parliaments, the rival proposals are complete bills. They are placed in some voting order; the first bill to win majority support is the voting outcome and voting stops; if all bills are rejected, the status quo prevails. Figure 10 shows a typical successive agenda. Clearly such an agenda is neither uniform nor complete. Indeed, it is a *partition* agenda in that, at every node, the set of remaining possible outcomes is partitioned in two non-overlapping sets, e.g., $\{b_k\}$ and $\{b_{k-1},...,b_1,q\}$, and there are no unchallenged alternatives. Given Farquharson's definition of sincere voting, the winning outcome is the first bill b_k to be voted on such that b_k is the majority winner in the set $\{b_k,...,b_1,q\}$; this implies that bills that come late in the voting order are greatly advantaged. An alternative definition of sincere voting under a successive agenda might be that voters vote in favor of any bill that they prefer to the status quo. Then the winning outcome is the first bill to be voted on that is majority preferred to q; this implies that bills that come early in the voting order have a great advantage (especially if the status quo is broadly unpopular). Under either assumption, the process clearly is not Condorcet consistent and is quite erratic in that it is highly dependent on the voting order. Realistically, the implication is that successive agendas encourage a substantial degree of 'strategic' voting, a topic to which we now turn.

Bibliographical Notes and Further Readings

The distinction between amendment and successive agendas, together with the tree representation of agendas, was initiated by Farquharson (1969). Ordeshook and Schwartz (1987), Miller (1965, Chapter 2), and Schwartz (2008) analyze parliamentary agendas in more detail; however, they differ in one respect that pertains to sincere voting, which issue is debated in Schwartz (2008, Appendix 2, and 2010) and Miller (2010a, 2010b). See Rasch (2000) for a discussion of parliamentary voting procedures in European parliaments. The effect of voting order on sincere (as well as strategic) voting outcomes under amendment and successive procedure is examined in Miller (1977), Bjurulf and Niemi (1982), Niemi and Gretlein (1982), Niemi and Rasch (1987), and Jung (1990).

10. Strategic Voting with Parliamentary Agendas

The discussion of 'strategyproofness' in 7.2 provided examples of incentives for strategic voting under various non-parliamentary voting rules. But that discussion did not (and could not) make definitive claims about what the winning outcome would be if all voters voted in a strategically optimal fashion. However, it turns out that, due to its sequential pairwise nature, strategic voting under parliamentary agendas can be definitively analyzed and strategic voting outcomes can be identified and compared with sincere voting outcomes.

As a warm-up exercise, let's consider the Powell amendment example once again. We have seen that, given Profile 10 and sincere voting, the final vote under an amendment agenda poses a choice between a bill for federal aid restricted to non-segregated schools and the status quo and that the bill is defeated. But there are at least two ways by which Northern Democrats might secure federal aid to education. One would be to persuade Representative Powell not to introduce his amendment. But the relevant possibility here is for Northern Democrats simply to vote contrary to their preferences and oppose the Powell Amendment on the first vote and thereby defeat it; the final choice is then between the unamended bill and no bill, which produces a victory for the bill and gives Northern Democrats an outcome that is their second, rather than third, preference.[43] Since Southern Democrats are now getting their first preference, they certainly have no incentive to engage in any counter-maneuver. While Republicans are now getting their third preference and certainly have an incentive to engage in a counter-maneuver, none is available.

Let us also consider a variant of this example in which Profile 10 is modified so that Republicans rank the unamended bill second and the amended bill third. Preferences are now single-peaked and *b* is the Condorcet winner. Thus, given sincere voting, the amendment loses on the first vote, and the unamended bill passes on the second vote. Opportunities for strategic voting exist but do not ultimately change the outcome. Taking sincere voting as the starting point, Republicans have an incentive to vote contrary to their preferences in support of the Powell amendment on the first vote; burdening the bill with the amendment leads to its defeat in the final vote, whereas it would otherwise pass, i.e., Republicans can strategically support a 'killer amendment' that they actually oppose.[44] But Northern Democratic supporters of the bill can counter this as before by strategically opposing the amendment. These two strategic moves counteract each other and leave the outcome unchanged.

We can now formalize and extend this logic of 'look ahead and reason back' that allows us to trace out the effects of strategic voting under parliamentary agendas. The first point to recognize is that, in order to vote strategically, voters must be able to anticipate the results of possible later votes, which implies that they know each other's preferences or, in any case, the relevant parts of the majority preference tournament. Consider once again Figure 7(a) showing the agenda tree for the original Powell amendment example. Given the majority preference tournament implied by Profile 10 and understanding that no voter has an incentive to vote other than sincerely at the final vote, we can determine the prospective outcomes of each possible final vote; these are indicated in Figure 7(a) by being underlined. Thus, we — and appropriately informed strategic voters who can anticipate that *b+a* would lose to *q* but *b* would beat *q* — understand that the first vote is effectively a choice between these two underlined *strategic equivalents q* and *b*, rather than between the two challenged alternatives *b+a* and *b* that sincere voters choose between. Since a majority of voters prefer *b* to *q*, *b* is the strategic equivalent at the initial node and is the strategic voting outcome.

This logic can be extended to any sequential pairwise agenda and majority preference tournament, though as the number of alternatives (and consequently the length of paths in the agenda

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43 While this strategem would certainly be advantageous to Northern Democrats in the immediate legislative context, it might also be costly to them in a broader electoral context, since it requires them to vote in a way that it likely to anger their liberal and/or African-American constituents.

44 Indeed, some might suspect that the modified profile was the 'true' profile in the 1950s and that Republicans in fact supported the Powell amendment only strategically.

tree) increases, it perhaps becomes less plausible that voters will or can actually make the required calculations. Recall that Figure 6 shows an arbitrary (and complex) agenda tree. Given the majority preference tournament shown in Figure 2, it also shows (again by underlining them) the strategic equivalents at every node. Since x_1 is the strategic equivalent at the initial node, it is the strategic voting outcome, reached as majority votes follow the short leftmost path to the only terminal node giving x_1 as the outcome.

Because every alternative is a possible voting outcome, i.e., is associated with at least one terminal node and, because as we move from any terminal node up to the initial node, such an alternative can be displaced as the strategic equivalent only by an alternative that beats it, it follows that every sequential pairwise voting rule is Condorcet consistent if voting is strategic (even if, as under successive procedure, it is not when voting is sincere). Moreover, since a top cycle alternative can be displaced as the strategic equivalent only by another top cycle alternative, only top cycle alternatives can be strategic voting outcomes.[45] However, which top cycle alternatives may become the strategic voting outcome depends on the particular structure of the agenda.

10.1 Amendment Agendas

Under an amendment agenda, every final vote pits some version of the bill against the status quo and voters have no reason to vote other than sincerely. Thus the strategic voting outcome must be either q or some version of the bill that beats q, i.e., some alternative in the win set $W(q)$. Suppose that some alternative x beats q; then x is the strategic equivalent at some final node and, as we move up the branch from the initial node to this final node, x can be displaced as the strategic equivalent only by some other alternative y that beat both x and q. And if x is displaced by y, y in turn can be displaced only by some z that beats y and x and q. And so forth. Thus q is the strategic voting outcome only if $W(q)$ is empty, i.e., q is the Condorcet winner; otherwise the strategic voting outcome belongs to the top cycle of the subtournament that includes only the alternatives in $W(q)$. We designate this set of alternatives $W^*(q)$.

Let us apply these considerations to the majority preference tournament in Figure 1 that represents Profile 7. If v is the status quo and enters the voting last, the strategic voting outcome is y because $W(v) = \{y,z\}$ and y beats z so $W^*(v) = \{y\}$. If x is the status quo, the strategic voting outcome is z because $W(x) = \{z,v\}$ and z beats v so $W^*(x) = \{z\}$. If y is the status quo, the strategic voting outcome is x because $W(y) = W^*(y) =\{x\}$. If z is the status quo, the strategic voting outcome is y because $W(z) = W^*(z) =\{y\}$. Note two contrasts between sincere and strategic voting outcomes. First, the latter tends to be less voting-order dependent. Second, not every alternative in the top cycle set is a potential strategic voting outcome; in the present example, v beats only x, so it can be the strategic voting outcome only if x is voted on last but, since z beats both v as well as x (i.e., z covers x), it is not the outcome even in this case. More generally, only alternatives in the uncovered set can be strategic voting outcomes under amendment agendas.

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[45] It may happen that the strategic equivalents at both nodes following a given node are the same alternative, in which case that alternative is also the strategic equivalent at the given node as well.

In this simple example based on Profile 7, $W^*(q)$ is a one-element set for every voting order. If this is not the case, the strategic voting outcome depends on which alternative in $W^*(q)$ enters the voting last. Suppose for example that the tournament that we have just examined is the top cycle set of some larger tournament. Then $\{x,y,z,v\}$ is also $W^*(q)$, and the strategic voting outcome depends on which alternative x, y, z, or v enters the voting last in just the way we examined above. More generally, if $W^*(q)$ includes alternative z and z is the last alternative in this top cycle to enter the voting, the strategic voting outcome is some alternative in the top cycle set of the intersection $W^*(q) \cap W(z)$. And so forth.

These considerations indicate that, while an alternative is favored by being placed late in the voting order if voting is sincere, the reverse is true if voting is strategic. Indeed, if alternative x is the strategic voting outcome under a given voting order, x is also the strategic outcome under any other voting order that is identical to the first with respect to the alternatives that follow x. Furthermore, if under a given amendment agenda the sincere and strategic voting outcomes are different, it can be shown that the latter precedes the former in the voting order and is majority preferred to it.

Since the status quo enters the voting last, it follows that, given there is no Condorcet winner and the status quo is not a Condorcet loser, it is easier to change the status quo if voting is strategic than if it is sincere, as the Powell Amendment example (based on Profile 10) illustrates.

10.2 Successive Agendas

Consider the typical successive agenda depicted in Figure 10. Clearly the strategic equivalent at the final node is b_1 if b_1 beats q and is q if q beats b_1; at the next-to-final node, it is b_2 if b_2 beats the strategic equivalent at the final node and otherwise is the latter; and so forth. This implies that strategic voting under a successive agenda works as if the last two alternatives in the voting order were paired for a sincere vote, the winner were paired with the third-to-last alternative, and so forth — that is, the process of tracing out strategic equivalents is equivalent to the process of tracing out sincere winners under ordinary committee procedure (or simple amendment agendas) when the voting order is reversed. Thus, given strategic voting, not only is successive procedure Condorcet consistent and only top cycle alternatives can become voting outcomes, but also every top cycle alternative may be the strategic voting outcome given some voting order. In addition, if alternative x is the strategic voting outcome under a given voting order, x is also the strategic outcome under any other voting order that is identical to the first with respect to the alternatives that follow x. Finally (as with amendment agendas), if under a given successive agenda the sincere and strategic voting outcomes are distinct alternatives, it can be shown that the latter precedes the former in the voting order but (unlike with amendment agendas) it need not be majority preferred to it.

10.3 Logrolling

We conclude this module by briefly reconsidering the example given by Profile 9. If the two issues X_1 and X_2 are voted on in turn and all voters vote sincerely, (x_1,x_2) is the voting outcome. Is the outcome different given strategic voting? Figure 10 shows the agenda tree and the strategic equivalents implied by Profile 9, indicating that (x_1,x_2) is also the strategic voting outcome. This indicates that strategic voting does not take account of incentives that voters with (somewhat) different preferences may have to cooperate with one another to advance interests they have in common. Here, voters 1 and 2 (who might actually be blocs of voters with similar preferences in the manner of the Powell Amendment example) have an incentive to form an explicit (majority) *coalition of minorities* (voter 2 constituting the minority on the first issue and voter 3 the minority on the second issue) to bring about the outcome that they both prefer to what otherwise would occur. Each can say to the other: "I will vote the way you want on the issue you care more about, if in return you vote the way I want on the issue I care more about'. Put otherwise, they are 'intense minorities' who can 'trade votes' or 'logroll' to bring about an outcome that includes each voter's preferred alternative on the issue he cares most about.

But such logrolling requires an explicit agreement between voters, not merely individual (or bloc) calculation in the manner of strategic voting. Moreover, each voter must be confident that the other will not renege on the agreement, despite the fact that each has a strong incentive to do since, if one voter reneges and the other does not, the outcome is the former's first preference and the latter's last preference. Furthermore, voter 1 has an incentive to break up a prospective agreement between voters 2 and 3 by, for example, offering an agreement on $(x_1,\bar{x}_2)$ to voter 2, as both prefer $(x_1,\bar{x}_2)$ to $(\bar{x}_1,\bar{x}_2)$. Indeed, the Condorcet cycle in Profile 9 implies that no agreement is stable against such efforts.

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Farquharson (1969) pioneered the analysis of strategic (which he called 'sophisticated') voting using 'sequential elimination of dominated strategies'. The method used here to identify strategic voting outcomes — characterized as 'looking forward and reasoning back', an expression adopted from Dixit and Nalebuff (1991) — is more technically known as 'backwards induction' and was formalized with respect to strategic voting by McKelvey and Niemi (1978), though it had been anticipated in earlier works. Gretlein (1983) showed that the two methods identify the same strategic voting outcomes. Miller (1980) showed that strategic voting outcomes under simple amendment procedure belong to the uncovered set; Shepsle and Weingast devised the 'sophisticated voting algorithm' to identify such outcomes, which Banks (1985) then employed to provide an exact characterization of the set possible strategic voting outcomes. Banks (1989) also examined strategic voting under two-stage amendment agendas. Denzau et al. (1985) observed that strategic voting may be inexpedient for legislators concerned about re-election. However, Austen-Smith (1987) shows that, given a plausible strategic agenda-building process, sincere and strategic voting may be equivalent.

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Figures

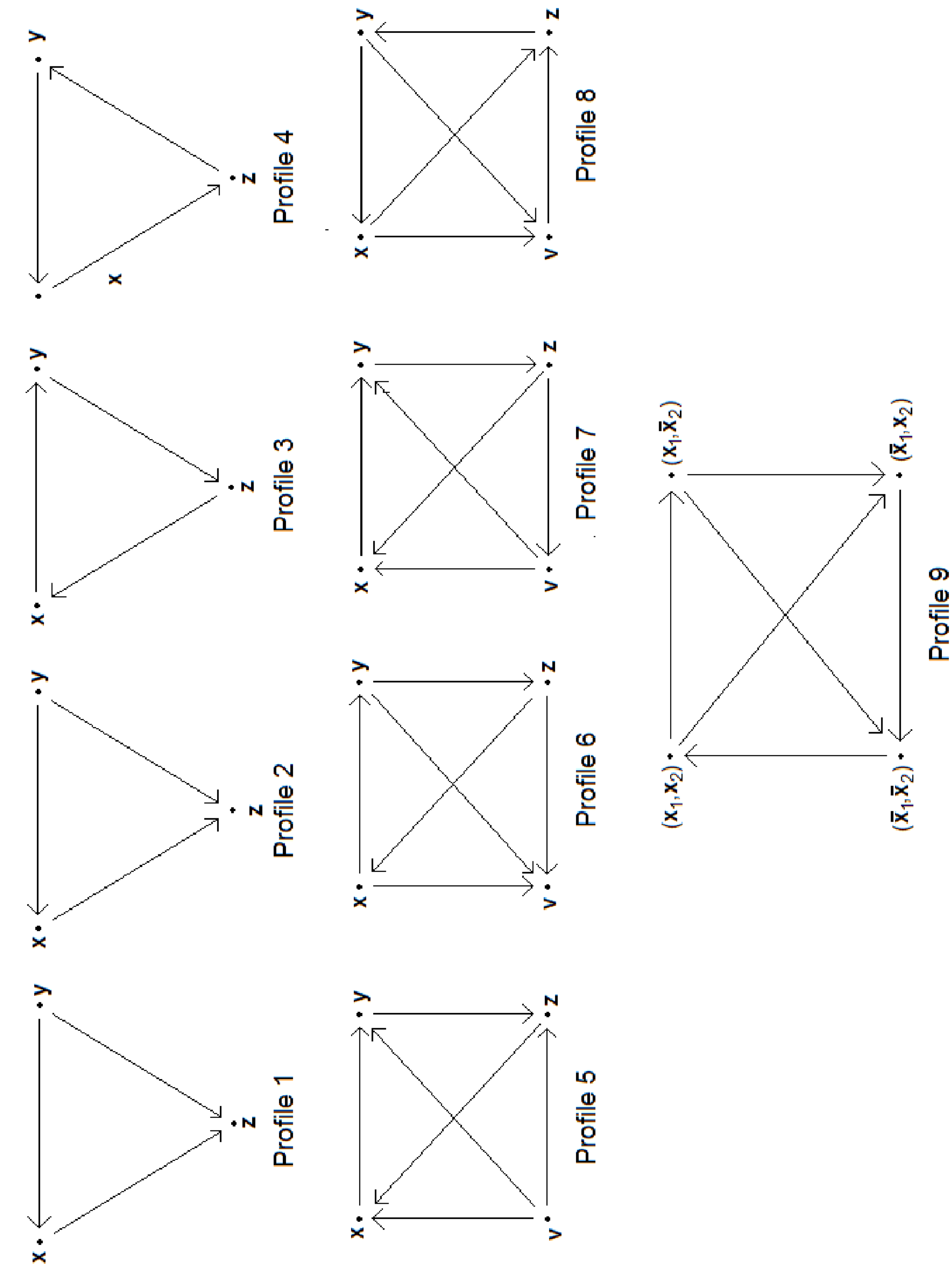

Figure 1 Tournaments Representing Profiles 1-9

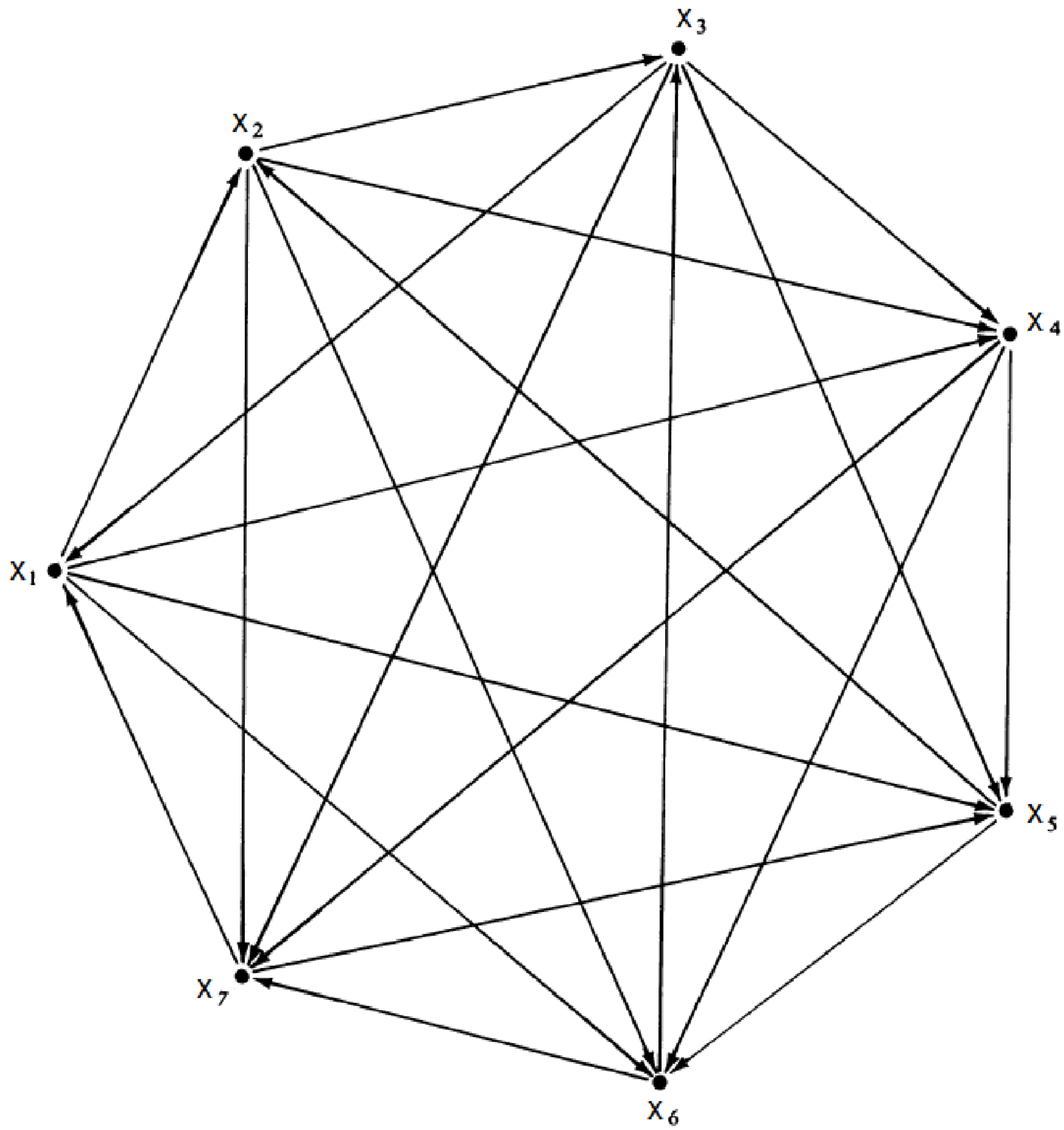

Figure 2 A Strong Tournament

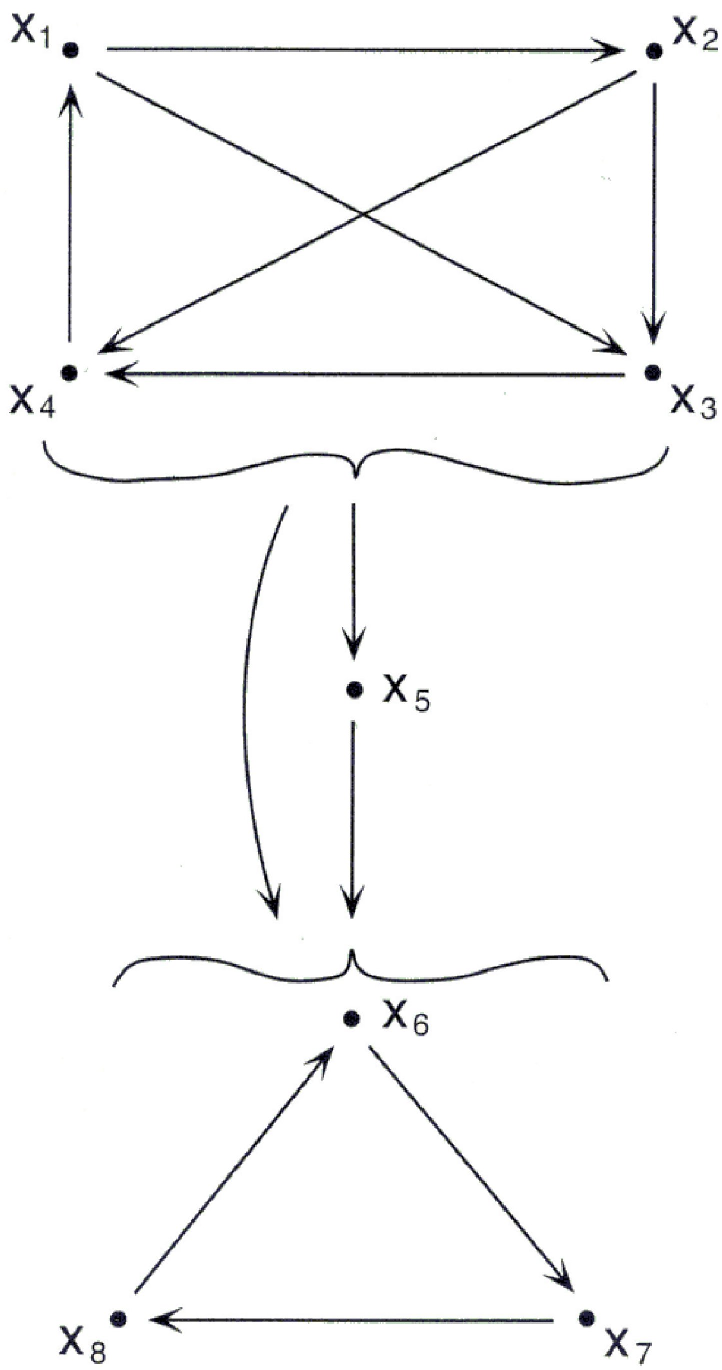

Figure 3 A Tournament Showing the Top Cycle Set

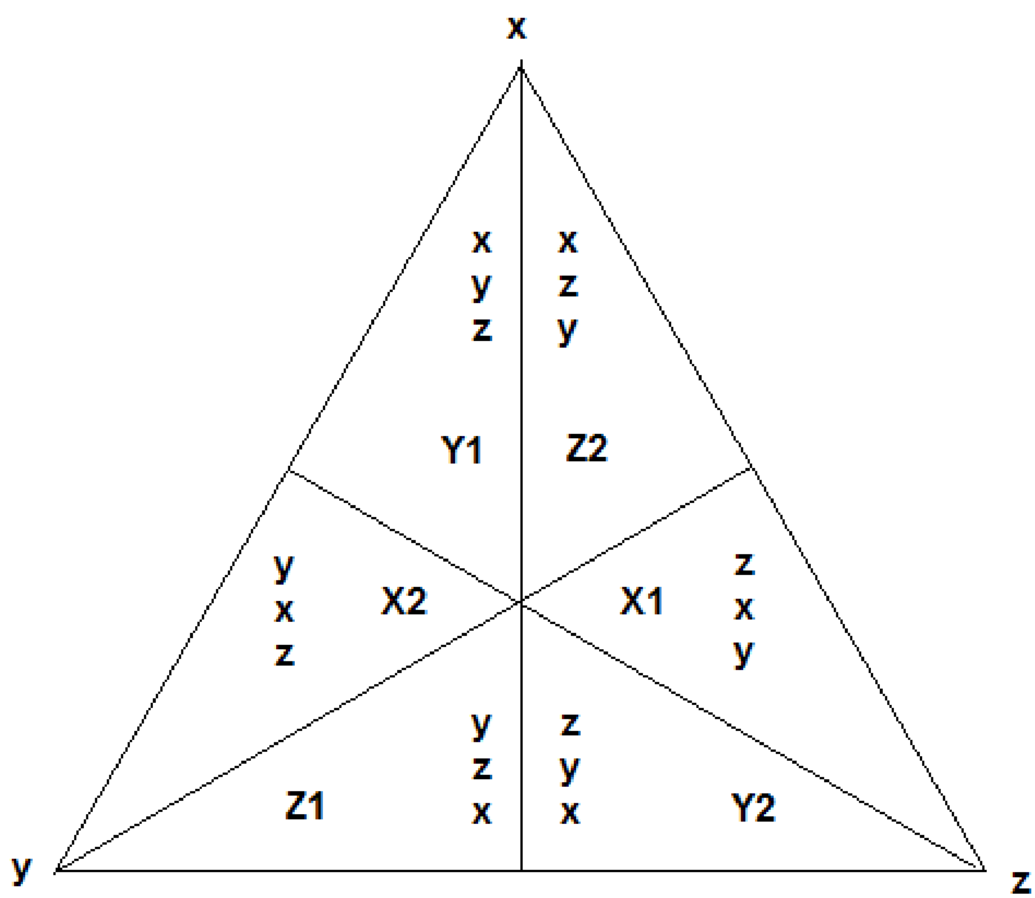

Figure 4 The Representation Triangle

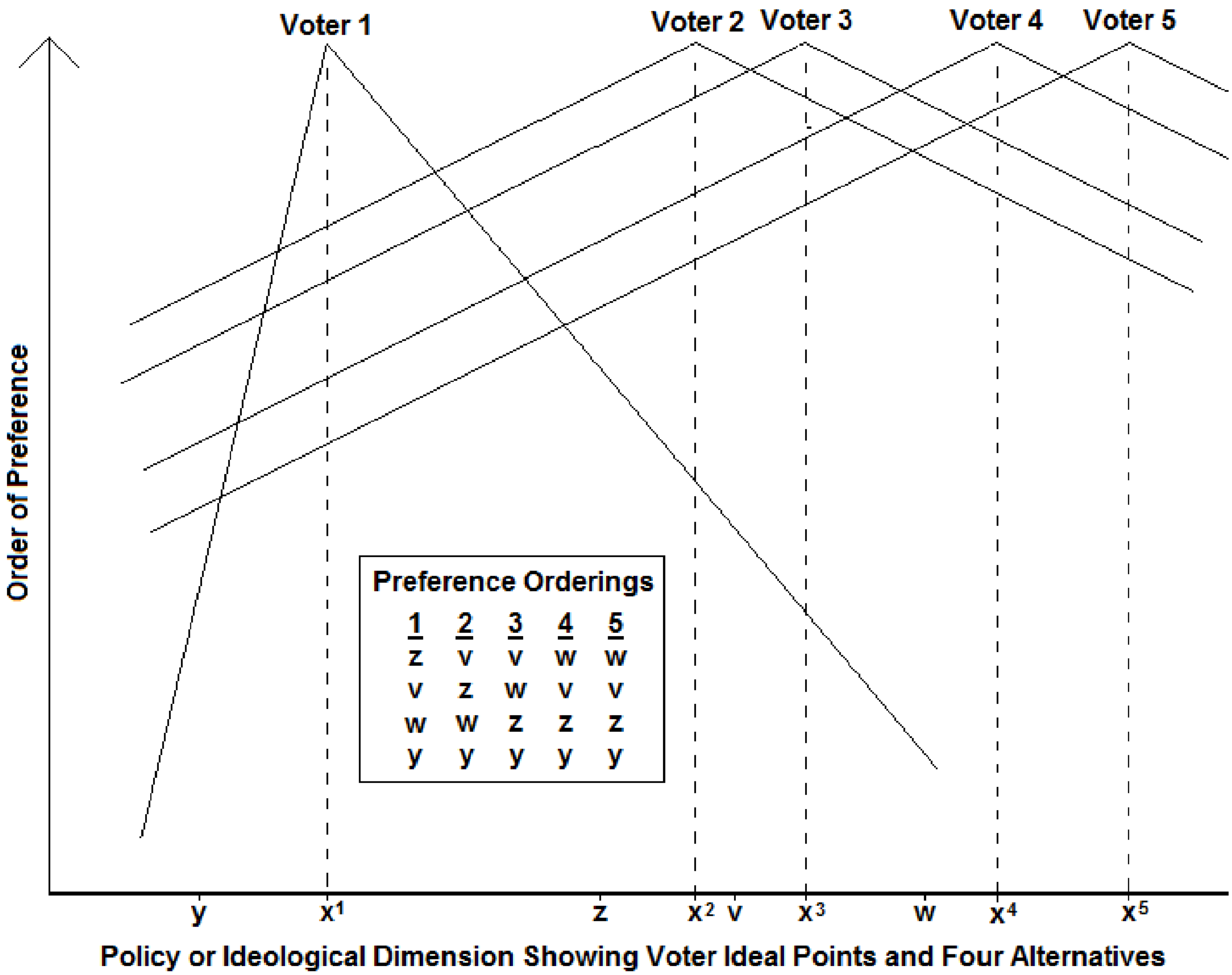

Figure 5 Single-Peaked Preferences

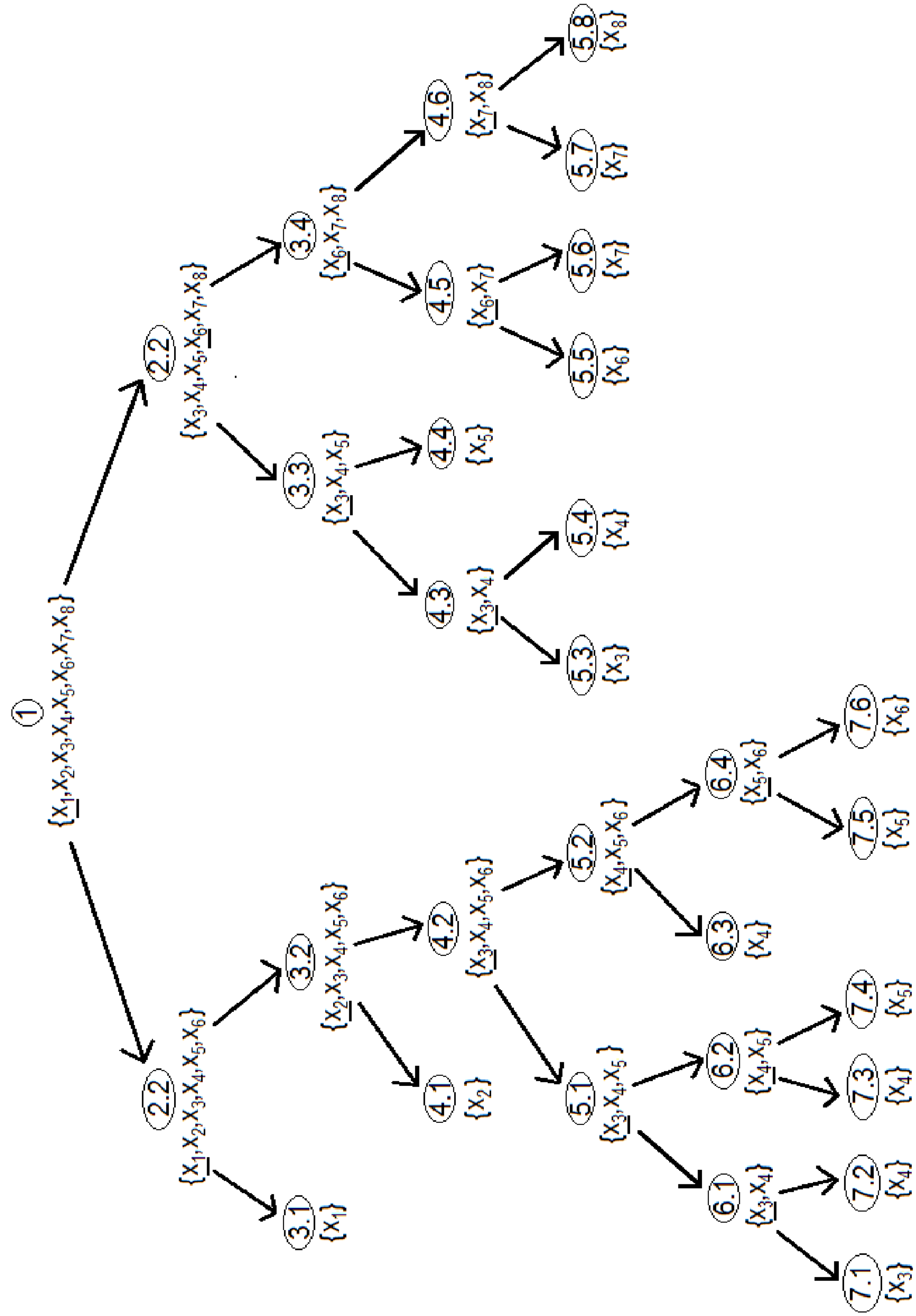

Figure 6 An Arbitrary Agenda Tree

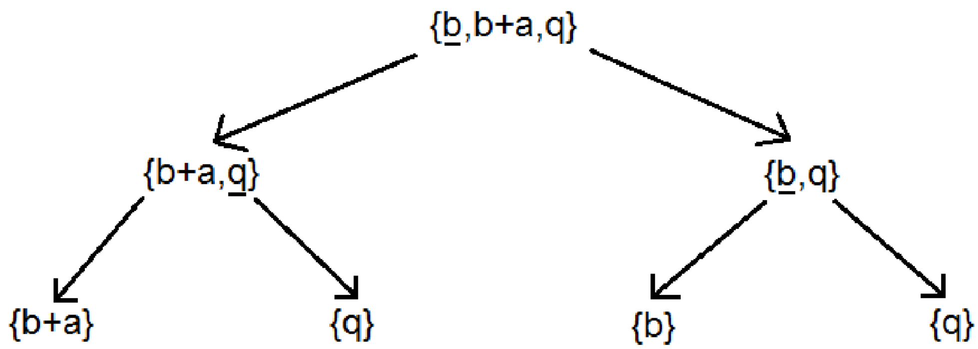

Figure 7(a) The Agenda Tree for the Powell Amendment

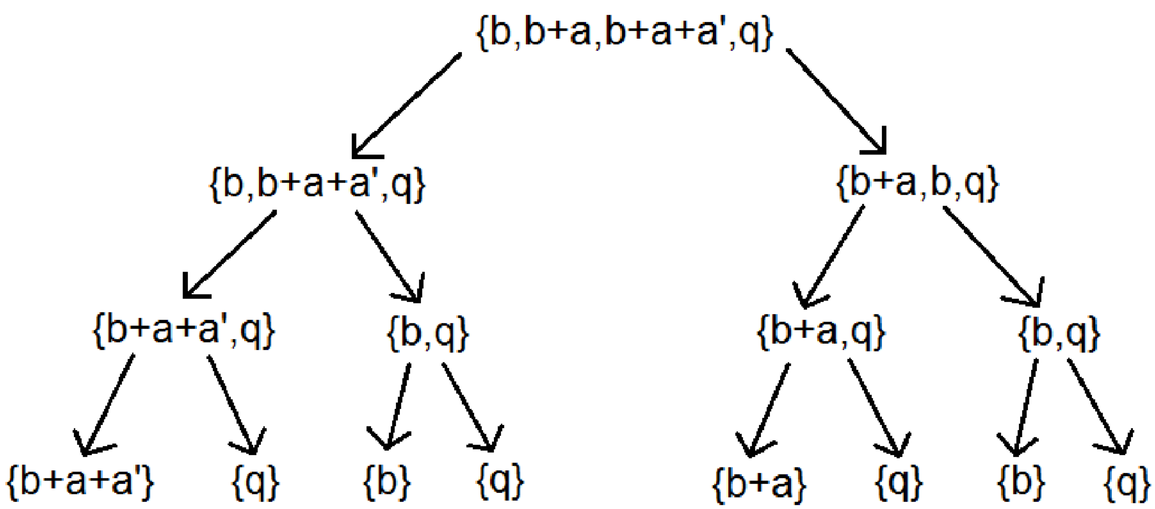

Figure 7(b) The Agenda Tree with an Amendment to the Powell Amendment

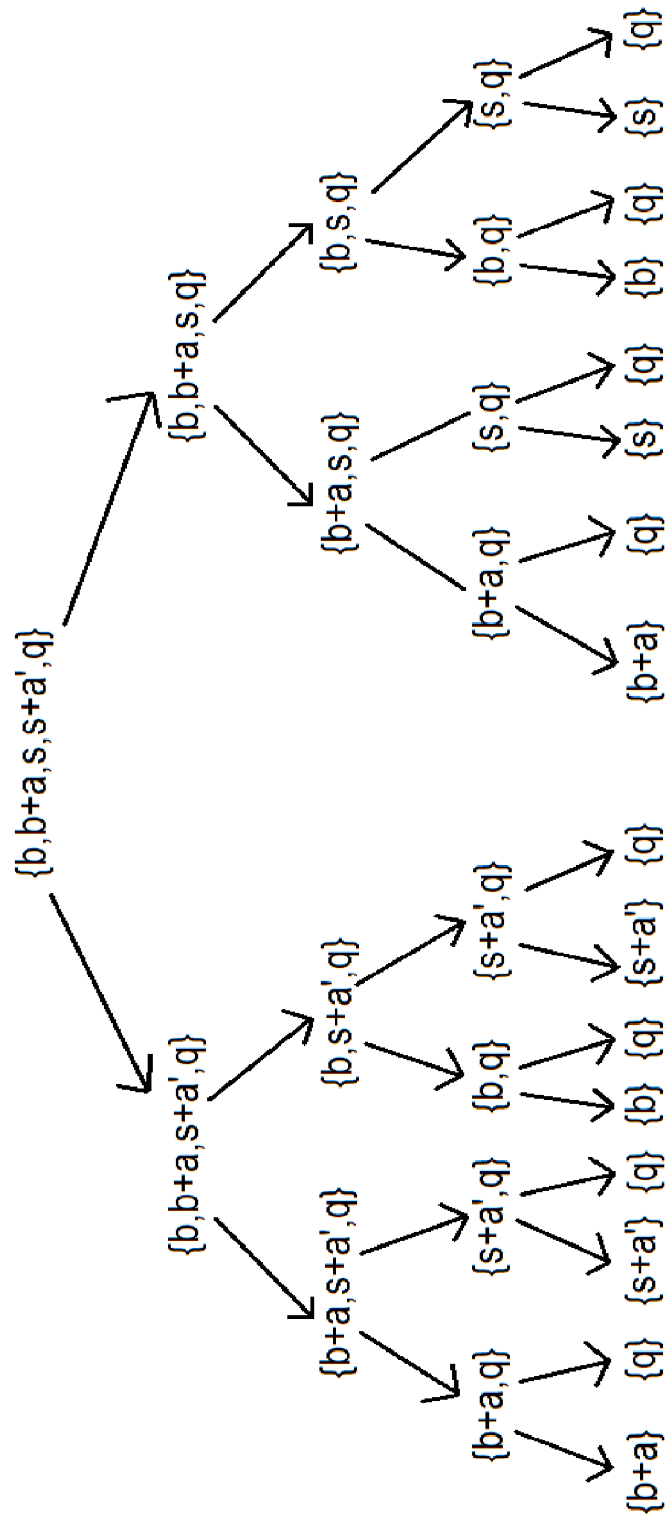

Figure 8 A Two-Stage Amendment Agenda

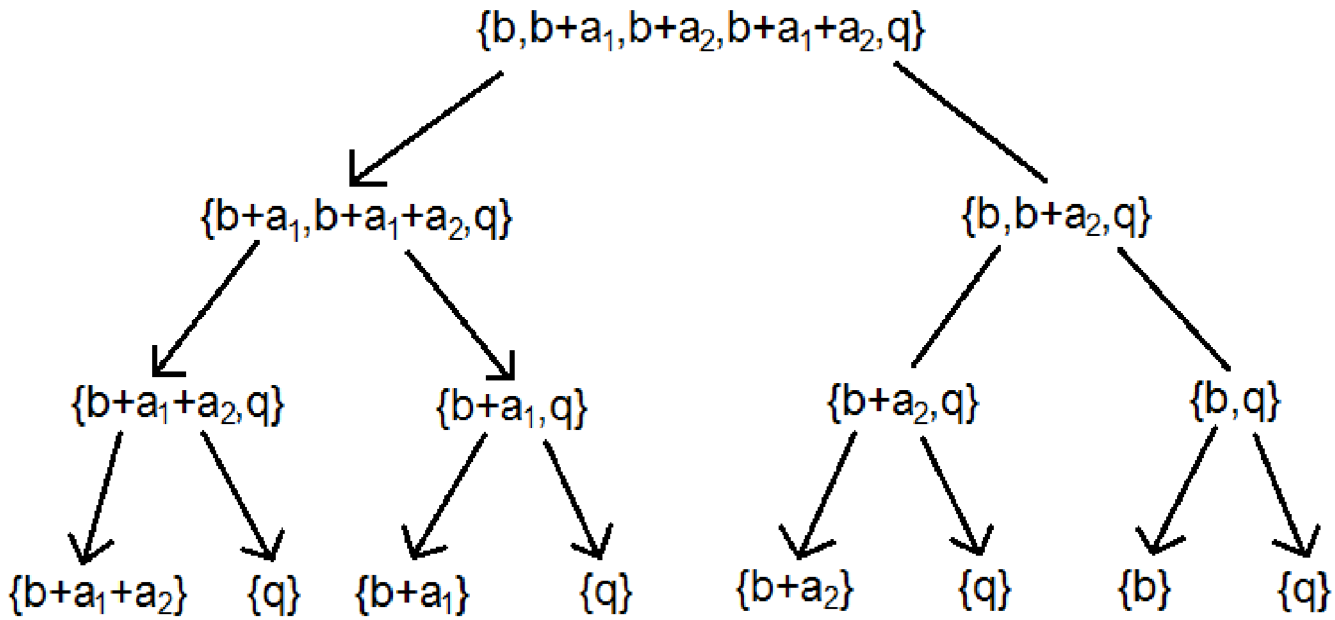

Figure 9 An Agenda with Two Compatible Amendments

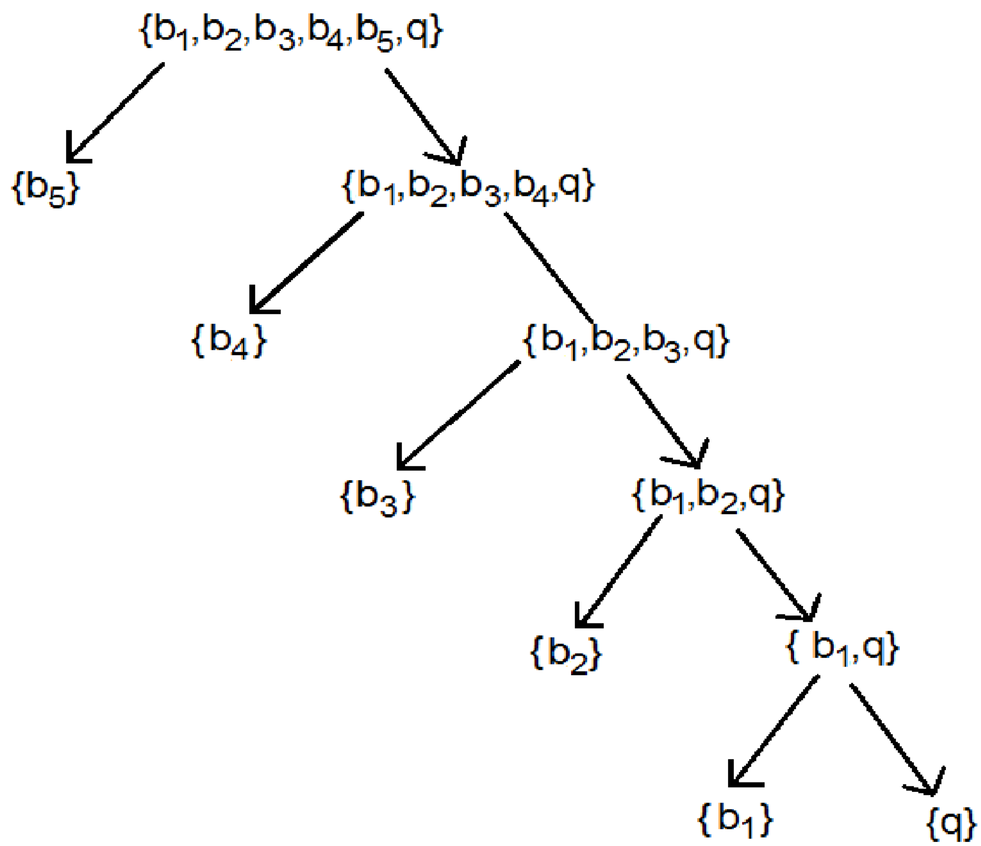

Figure 10 A Successive Agenda