THE HOUSE SIZE EFFECT AND THE REFERENDUM PARADOX IN U.S. PRESIDENTIAL ELECTIONS

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Abstract

Barthélémy, Martin, and Piggins (2014), extending the work of Neubauer and Zeitlin (2003), show that some U.S. presidential elections are subject to a 'House size effect' in that the winner of the election, i.e., the candidate who wins a majority of electoral votes, depends on the size of the House of Representatives. The conditions for the effect relate to the number of 'Senate' versus 'House' electoral votes won by each candidate, but the relationship is not straightforward due to 'locally chaotic' effects in the apportionment of House seats among the states as House size changes. Clearly a Presidential election that is subject to the House size effect exhibits the referendum paradox, i.e., the electoral vote winner is the popular vote loser, for some House sizes but not for others.

Keywords: U.S. presidential elections; Electoral College; House size effect; referendum paradox

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1. Introduction

The U.S. Constitution fixes some features of the Electoral College system for electing Presidents, including the stipulations that each state has electoral voters equal in number to its total representation in Congress and that each state has two Senators. However, the Constitution does not fix the size of the House of Representatives, which can be changed by statute.

Some years ago, Neubauer and Zeitlin (2003) observed that, if the ratio of House seats to population that existed in 1940 had been maintained in 1990 (the census and apportionment that governed the 2000 election), the House would have had about 830 members. Moreover, Neubauer and Zeitlin calculated that, given a House of 830 members (and the existing Hill-Huntington apportionment method), Gore would have won the 2000 election with 471 electoral votes to 463 for Bush. More generally, they show that Bush would have won the election with any House size through 490, Gore would have won with any House size of 598 or greater (except for an electoral vote tie with a House of 655 members), and outcomes for House sizes between 491 and 597 would have fluctuated in a chaotic fashion between narrow Bush victories, narrow Gore victories, and perfect ties.

Using what they call 'representation graphs,' Barthélémy, Martin, and Piggins (2014) have recently expanded Neubauer and Zeitlin's analysis in several ways: first, they examine additional presidential elections; second, they consider the effects of additional apportionment methods; third, they allow for other Senate sizes and House representation floors (the Constitution guarantees every state one House seat); and fourth, they consider the phenomenon of the 'referendum paradox' (or 'election inversion,' in the language of Miller, 2012) that was a notable feature of the 2000 election, i.e., the fact that Bush won a majority of electoral votes while Gore won a plurality of popular votes.

Somewhat surprisingly, neither Neubauer and Zeitlin nor Barthélémy et al. in the original version of their paper sought to identify the logical conditions that determine whether what we may call the 'House size effect,' as was present in the 2000 election, can arise. The purpose of this note is to identify these conditions and to draw out their implications.

2. Senate versus House Electoral Votes and the House Size Effect

In the language of Madison's Federalist 39, the apportionment of electoral votes is a compromise between the 'federal' principle of state equality embodied in the Senate and the 'national' principle of state representation according to population embodied in the House of Representatives. The ratio of House to Senate size determines the weight given to the 'national' versus 'federal' principles in the Electoral College. Given a fixed number of states, the size of the House determines this ratio; an increase in the size of the House makes the Electoral College more 'national' and the allocation of electoral votes among the states more proportional to their populations.

For purposes of this analysis, we deem a presidential election to be a two-candidate affair, specified by the popular vote for each candidate in each state. It is assumed that the candidate who wins the popular vote of a state wins all of its electoral votes and the District of Columbia is treated as if it were a state. A candidate is said to win two 'Senate' electoral votes for each state he carries, the remainder being 'House' electoral votes.

Typically the federal and national principles give the same verdict — that is, one candidate carries a majority of states and therefore wins a majority of Senate electoral votes and the same candidate wins a majority of House electoral votes. But sometimes the two principles give conflicting verdicts. For example, in the 2000 election Bush carried 30 states and Gore carried 21, giving Bush a margin of 18 with respect to Senate electoral votes, while Gore won 225 House electoral votes to 211 for Bush, giving Gore a margin of 14, so Bush won by an overall margin of four electoral votes.[1] However, to take if the House size had been larger, Gore's House electoral vote margin would have been increased in about the same proportion and, with a sufficiently larger House, Gore would have won an overall electoral vote majority.

We say that a presidential election is subject to the *House size effect* if the winner of the election, i.e., the candidate who wins a majority of electoral votes, depends on the size of the House. The previous discussion appears to support the following proposition.

Proposition A. A presidential election is subject the House size effect if and only if:

(1A) one candidate (say A) wins a majority of the Senate electoral votes, *and*
(2A) the other candidate (say B) wins a majority of the House electoral votes.

As a summary of the historical record of presidential elections since 1828, Proposition A in fact holds up. The top rows of Table 1 show the six presidential elections that meet conditions (1A) and (2A); all these elections are subject to the House size effect and no others are. We will discuss these elections in more detail in Section 5.

But as a theoretical proposition, Proposition A does not (quite) hold up. This is because the apportionment of House seats is an unavoidably quirky matter, given that each state must be allocated a whole — and relatively small — number of House seats, and whole numbers cannot be apportioned in a way that is perfectly proportional to the 'apportionment population' of each state.[2] This implies that whether a candidate wins a majority of House electoral votes may itself depend on House size.

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[1] Gore lost one electoral vote to a faithless elector. The number of House electoral votes is one greater than the actual House size (i.e., 436 rather than 435) because DC is deemed to be a state.

[2] The U.S. Census Bureau determines different 'official' populations of states for different purposes; 'apportionment population' is the official population for apportionment purposes. While the Census Bureau does not report an apportionment population for the District of Columbia (because DC is not included in the apportionment of House seats), such a population is here defined in the same manner as for states.

3. Perfect Apportionment and the House Size Effect

For theoretical purposes, the following concept allows us to sidestep this apportionment problem: the House of Representatives is (hypothetically) *perfectly apportioned* if each state is allocated a *quota* of House seats that is perfectly proportional to its apportionment population.[3] Of corse, such a quota is almost never a whole number. This concept allows us to state the following logically correct necessary and sufficient conditions for the House size effect.

Proposition 1. If the House were perfectly apportioned, a Presidential election would be subject to the House size effect if and only if:

(1) one candidate (say A) carries a majority of states, *and*
(2) the other candidate (say B) carries states that collectively hold a majority of the total apportionment population.

Let S_A and S_B be the number of Senate electoral votes won by the two candidates, let H be the size of the House, and let p_A and p_B be the proportion of the total apportionment population held by states carried by the two candidates; then, given perfect apportionment, candidates A and B win $S_A + (p_A \times H)$ and $S_B + (p_B \times H)$ electoral votes respectively. Given perfect apportionment, the number of (House and total) electoral votes won by each candidate is a linear function of H, and one candidate's (House and total) electoral vote margin over the other is therefore also a linear function of H.

To establish sufficiency, suppose that $S_A > S_B$ and $p_B > p_A$. Then candidate A wins the election when the House is of minimum size (which must be zero, as a guaranteed floor is incompatible with perfect apportionment), since in this event each state has the same number of electoral votes and candidate A wins a majority of them. But candidate B wins a majority of House electoral votes and wins the election if $S_B + (p_B \times H) > S_A + (p_A \times H)$. Rearranging terms, this expression becomes $H > (S_A - S_B) / (p_B - p_A)$, which, given any S_A and p_B, must be true for sufficiently large H. Thus (1) and (2) together imply the House size effect.

To establish necessity, suppose that $S_A \geq S_B$ and $p_A \geq p_B$ (or $S_B \geq S_A$ and $p_B \geq p_A$). Then $S_A + (p_A \times H) \geq S_B + (p_B \times H)$ [or $S_B + (p_B \times H) \geq S_A + (p_A \times H)$] for all H. If the inequalities are strict, the same candidate wins for all House sizes. If $S_A = S_B$, a House of minimum size produces an electoral vote tie but candidate A wins with a larger House (which we may call a *partial* House size effect) unless $p_A = p_B$, in which case there is an electoral vote tie for all House sizes.

The equality $H = (S_A - S_B) / (p_B - p_A)$ specifies the *crossover* House size that occurs when candidate A's margin with respect to Senate electoral votes is precisely balanced by candidate B's

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[3] 'Perfect apportionment' is also achieved by what Barthélémy et al. (2014, p. 4?) call 'limit apportionment,' which occurs when House and population size are equal, making perfect apportionment possible in whole numbers. 'Perfect apportionment' as defined here differs from the similar term used in Miller (2012), which pertains to perfect proportionality between electoral votes and popular votes (rather than between House seats and population).

margin with respect to perfectly apportioned House electoral votes, producing an electoral vote tie. A smaller House produces an electoral vote majority for candidate A, a larger House a majority for B.

Corollary 1.1. If the House were perfectly apportioned and a Presidential election is subject to House size effects, the crossover House size would be H = $(S_A - S_B) / (p_B - p_A)$.

By way of illustration, in 2000 Bush won 30 states to Gore's 21 and Gore carried states with 51.6834% of the apportionment population to Bush's 48.3166%, so perfect apportionment implies a crossover House size of $(60 - 42) / (0.516834 - 0.483166) \approx 534.64$.[4]

4. Apportionment Methods and the House Size Effect

Since House seats are not, and cannot be, perfectly apportioned among the states, we must now confront the apportionment problem.[5] There are many different apportionment methods, four of which have actually been used to apportion House seats and are briefly described in Barthélémy et al. (2014), and which may produce (slightly) different allocations of House seats for a given House size and profile of state populations.

Moreover, given any such method, candidate B's margin with respect to House electoral votes generally increases with House size (given $p_B > p_A$), but this margin cannot increase in a linear, or even monotonic, fashion. Suppose that the House is a given size and is then increased by one seat. The additional seat will be awarded to the state with the strongest claim to it (according to the apportionment method in use) but that state may well have been carried by A, in which event B's electoral vote margin decreases rather than increases.[6] As Neubauer and Zeitlin (2003) observe, the effect of a small increase House size on candidate B's electoral vote margin over A 'depends intricately on population sizes [and the particular apportionment method] and cannot be discerned

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[4] This is near the midpoint of Neubauer and Zeitlin's (2003) chaotic range noted earlier. With the actual House size of 436, perfect apportionment implies that Gore would have won 225.34 House electoral votes and Bush 210.66; they actually won 225 and 211. (However, the House electoral votes actually won by the candidates are not always equal to those won under perfect apportionment rounded to the nearest whole number.)

[5] Balinski and Young (2001) provide the definitive treatment of the apportionment problem as it applies to the U.S. House of Representatives.

[6] Indeed, if the apportionment formula is not *house monotone* (Balinski and Young, 1982, pp. 38-39), a one-seat increase in House size may not only award the extra seat to a state carried by A but also take a House seat away from a state carried by B and award it to a second state carried by A. The Hamilton (quota) method used during much of the nineteenth century has this paradoxical property, but all other (divisor) methods discussed by Barthélémy et al. (2014), including the existing Hill-Huntington method, are house monotone. In my view and contrary to that expressed by Neubauer and Zeitlin (2003), the 'non-monotonicity of electoral vote margins with respect to House size' is of a quite different character from the non-monotonicity of the Hamilton method: the former, but not the latter, is inevitable for any apportionment method and, on due consideration, should not be surprising — to avoid it, candidate B would have to be awarded every House electoral vote beyond those entailed by the minimum House size.

a priori.' We may characterize the relationship as 'locally chaotic'; the character of the relationship is vividly illustrated in Neubauer and Zeitlin's Graphs 1 and 2.[7]

This local chaos has several implications. First, as Neubauer and Zeitlin show for 2000, candidate A wins an overall majority of electoral votes up to some threshold House size, candidate B wins an overall majority of electoral votes above a second higher threshold House size, and between these two thresholds there is seemingly random fluctuation between (narrow) electoral vote victories for A, (narrow) victories for B, and electoral vote ties. This generalization holds for all apportionment methods, though the details vary by the method.

Second, local chaos implies that, even if the 'federal' principle is given no weight (i.e., if there are only House electoral votes), or if the candidates carry equal numbers of states, an election may be subject to a (full, not partial) House size effect. This would be the case if p_B barely exceeds 0.5; then candidate A would likely win at some small and medium House sizes, though B would win consistently at sufficiently large sizes.[8] More generally, while every apportionment method guarantees that a more populous state is awarded at least as many House seats as a less populous state, no apportionment formula can guarantee that a *set of states* with a greater population (such as those carried by candidate B) is awarded at least as many House seats as another set of states with a smaller population (such as those carried by candidate A).[9]

Local chaos resulting from apportionment into whole numbers raises the question of how Proposition 1 needs to be modified to take account of apportionment methods. Conditions (1) and (2) remain jointly sufficient for the House size effect but the complexities introduced by local chaos mean that their conjunction is no longer (quite) necessary.

Proposition 2. Given any method of apportionment method into whole numbers, a Presidential election is subject to the House size effect if:

(1) one candidate (say A) carries a majority of states, *and*
(2) the other candidate (say B) carries states that collectively hold a majority of the total apportionment population.

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[7] Local chaos also shows up in the representation graphs of Barthélémy et al. (2014) but in an attenuated fashion, because their charts plot electoral votes in five-unit, rather than one-unit, intervals of House size.

[8] As Neubauer and Zeitlin's (2003) graphs indicate, the disproportionality in House electoral votes (relative to that entailed by perfect apportionment) that local chaos can bring about — not typically but at particular House sizes — can be surprisingly large. For example, their Graph 2 shows that, given a House size of 523 in the 2000 election, Bush would have won a majority 264 House electoral votes, as opposed to a minority of about 252.7 under perfect apportionment.

[9] In like manner, Miller (2013) demonstrates that an 'election inversion,' such that a coalition of parties supported by a majority of voters may win fewer seats in parliament than the complementary coalition supported by a minority of voters, is possible under even the purest type of proportional representation.

As before, candidate A wins if the House is of minimum size (i.e., either zero or each state getting only the guaranteed floor), while candidate B wins if the House is sufficiently large that it not only satisfies the expression $H > (S_A - S_B) / (p_B - p_A)$ but also overcomes (i) any bias in the apportionment method that may be adverse to B and (ii) any effects of local chaos.[10] On the other hand, suppose that $S_A > S_B$ and $p_A > p_B$; if S_A exceeds S_B only slightly, there may be some (small or modest) House sizes at which candidate B wins more House electoral votes than A, due to the effects of bias and/or local chaos and sufficient for an overall electoral vote majority. To take one example, suppose that candidate A carries just one more state than B and, though A carries states with a majority of the apportionment population, B carries the three largest states. Then A wins with a minimum, or minimum plus one, House size; there is an electoral vote tie at the minimum plus two size; and B wins with a House size of minimum plus three (but A wins consistently at sufficiently large House sizes). Thus we have what we may call a 'local' House size effect. And if S_A and p_A exceed S_B and p_B only slightly, the example cited in footnote 8 indicates that candidate B may win sufficiently more House electoral votes than A at particular House sizes well above the minimum size to win an overall electoral vote majority, also producing a 'local' House size effect.

In most cases, the formula in Corollary 1.1 produces a reasonable estimate of the 'expected' crossover House size under any actual apportionment method, but local chaos produces (as Neubauer and Zeitlin document for 2000) a range of House sizes around this expected size in which the winner is unpredictable (except by specifying the apportionment method and laboriously calculating the apportionment of seats for each House size in this range). However, if the 'expected' crossover size is quite small, special account needs to be taken of the guarantee of at least House seat for each state (or whatever the floor may be).

5. The House Size Effect, the Referendum Paradox, and Historical Presidential Elections

While we have thus far taken no account of which candidate is the national popular vote winner, the following proposition is essentially self-evident.

Proposition 3. A Presidential election that is subject to the House size effect exhibits the referendum paradox for some House sizes but not for others; one that is not subject to the House size effect either exhibits the referendum paradox for all House sizes or for none.

Regardless of House size, we have the same presidential election and the same popular vote winner (whether A or B) so, if an election is subject to the House size effect, a referendum paradox must exist given either a sufficiently small House size (such that A wins) or a sufficiently large House size (such that B wins). On the other hand, an election that is not subject to the House size effect

[10] The analysis of Barthélémy et al. (2014, p. 4?) shows that 'all representation graphs will converge toward the limit apportionment as [House size] increases.' This is because, when House size equals the size of the national population, all apportionment methods give perfect apportionment, eliminating both bias and local chaos. Short of this limit, the Jefferson and Adams methods are biased toward small and large states respectively, while the Hill-Huntington method is slightly biased toward small states and the Hamilton and Webster methods appear to be unbiased. If one candidate carries relatively many small states and the other mostly large states, these biases will have a (small) effect on the number of House electoral votes won by each candidate.

may or may not exhibit the referendum paradox, but if such an election does (or does not) exhibit the paradox with some House size, it exhibits it (or not) with all House sizes — in the language of Barthélémy et al. (2014), the paradox (or its absence) is 'entrenched.'[11]

We now briefly discuss the six historical presidential elections shown in Table 1 to be clearly subject to the House size effect, four of which were identified and discussed by Barthélémy et al. (2014) and two of which (1960 and 1860) were not. The 1960 and 1860 elections are complicated by the fact neither was a strictly two-candidate affair (which evidently is why Barthélémy et al. [2014] did not examine them), and 1860 is included only on the counterfactual assumption that Lincoln faced a united opposition. It can be checked than in each of these six elections, the candidate who won the most House electoral votes also won states with a majority of the apportionment population (and therefore would also have won the most House electoral votes under perfect apportionment). Indeed, in each election the actual House electoral vote hardly differed from that resulting from perfect apportionment.[12]

Identifying the candidates as A and B in the manner of Propositions 1 and 2, if candidate A is the electoral vote winner, a *larger* House would make candidate B the winner, while if candidate B is the electoral vote winner, a *smaller* House would make candidate A the winner.

Crosstabulating electoral voter winner status with popular vote winner status, we can classify the six historical presidential elections clearly subject to the (full) House size effect into four categories:

1. If candidate A is both the popular vote and electoral vote winner, there is no referendum paradox with the actual House size but a larger one would make candidate B the electoral vote winner and thereby create a referendum paradox. The 1916 election falls into this category, with Wilson (Dem) as candidate A and Hughes (Rep) as candidate B.

2. If candidate A is the popular vote winner and candidate B is the electoral vote winner, there is a referendum paradox with the actual House size but a smaller one would make candidate A the electoral vote winner and thereby remove the referendum paradox. The counterfactual 1860 election falls into this category, with the united opposition (Dem) as candidate A and Lincoln (Rep) as candidate B.

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[11] Note that candidate B, who carries states with a majority of the apportionment population, need not be the popular vote winner, since B may carry these states by small margins while A carries the remaining states by large margins. Moreover, B may carry most or even all of the smallest states. Thus Neubauer and Zeitlin's (2003, p. 722) remarks that 'since Gore won the popular vote, it is hence not surprising that he would have won the election for large House sizes' and that 'the reason Bush would have won with small House sizes . . . [is that Bush did especially well in small states and] smaller states have a relatively larger percentage of members in the EC' are off-the-mark. Gore would have won with larger House sizes simply because of condition (2) in Proposition 2, not because he was the popular vote winner, and Bush would have won with smaller House sizes simply because of condition (1), not because he won most of the smaller states.

[12] While this is typically true, it is not necessarily true; recall footnote 7.

3. If candidate B is the popular vote winner and candidate A is the electoral vote winner, there is a referendum paradox with the actual House size but a larger one would make candidate B the electoral vote winner and thereby remove the referendum paradox. The 2000 and 1876 elections fall into this category, with Bush and Hayes (both Rep) as candidate A and Gore and Tilden (both Dem) as candidate B.

4. If candidate B is both the popular vote winner and electoral vote winner, there is no referendum paradox with the actual House size but a smaller one would make candidate A the electoral vote winner and thereby create a referendum paradox. The 1976 and (probably) 1960 elections fall into this category, with Ford and Nixon (both Rep) as candidate A and Carter and Kennedy (both Dem) as candidate B.

The particulars of the 2000 election have already been discussed. We will briefly discuss the particulars of the others in reverse chronological order.

The 1976 election was subject to the House effect but, in contrast to 2000, the crossover House size was smaller, rather than larger, than the actual size — indeed, much smaller (about 41, given perfect apportionment), since Carter (candidate B) had a large advantage with respect to the population of states carried. But as House size gets smaller, the constitutional requirement that every state is entitled to at least one House seat has increasing bite and becomes incompatible with the assumption of perfect apportionment. Given the floor of one seat for every state, even a House of minimum size (51) would have produced an electoral majority for Ford of 81-72, and it is evident that somewhat larger House sizes would also produce Ford victories. A more accurate approximation that allows for this floor can be made by supposing, in effect, that each state has three Senate electoral votes, subtracting 51 from whatever House size would otherwise be, and maintaining Carter's advantage with respect to this smaller number of House seats. This produces a crossover size of about 113.[13]

In 1960, an anti-civil-rights 'unpledged elector' slate (whose votes were ultimately cast for Byrd) carried Mississippi; moreover, Alabama electors were individually elected, resulting in a split electoral vote between Kennedy and other unpledged electors (who also cast their votes for Byrd). Table 1 credits Kennedy with carrying Alabama and winning all of its electoral votes and sets Mississippi's electoral votes aside, but any accounting of the electoral votes of these two states leaves 1960 as an election subject to House size effects. As in 1976, the crossover House size was smaller than the actual size. A sufficiently smaller House size would have deprived Kennedy of his electoral vote majority and deadlocked the Electoral College (given Mississippi's votes for Byrd), while a still smaller size would have given Nixon the majority. Given Kennedy's large advantage with respect to percent of population in states carried, perfect apportionment implies that even the first crossover size is very small (about 29), so an adjustment would have to be made to allow for the guaranteed floor, in the same manner as for 1976. While the table reflects the fact that Kennedy is generally deemed to be the popular vote winner in 1960, the situation in Alabama presents complexities in establishing

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[13] The calculations of Barthélémy et al. (2014), directly applying the Hill-Huntington apportionment formula as reflected in their Figure 6, suggest a still larger figure.

the national popular vote (see Gaines, 2001). But, given the election was subject to the House size effect, it is clear that a referendum paradox existed at some House size however these complexities are resolved.

In 1916, Wilson carried a large majority of the states, while Hughes carried states with a bare majority of the apportionment population. A sufficiently large House would have produced an electoral vote majority for Hughes (and therefore a referendum paradox), but the crossover House size is extremely large, being on the order of 13,000.[14]

At first blush, the case of 1876 may seem a bit puzzling. Under perfect apportionment, the crossover House size was about 222, smaller than the actual House size of 293. Yet Tilden was candidate B, who would benefit from a larger House. In fact, Tilden lost by one electoral vote, not because the House was clearly too small but because of local chaos, the actual House size of 293 being near the upper end of the chaotic range; other House sizes, smaller as well as larger, would have given him an overall electoral vote majority.[15] Of course, a House size significantly larger than 293 would have guaranteed Tilden an overall electoral vote majority.

In 1860, Lincoln was the candidate of the recently formed Republican Party. The Democratic Party split into Northern and Southern wings, each with its own candidate (Douglas and Breckinridge, respectively). Moreover, the Southern remnant of the old Whig Party nominated a fourth candidate (Bell). Following Miller (2012), Table 1 turns 1860 into a strictly two-candidate election by assuming a united anti-Lincoln opposition (labelled ‘Dem’ in Table 1) that captures all the popular votes cast for Douglas, Breckinridge, and Bell. While this united opposition would have won over 60% of the popular vote, Lincoln would have retained a (somewhat reduced) electoral vote majority, so this counterfactual election would have produced the most massive referendum paradox in U.S. presidential election history. But because the anti-Lincoln opposition would have carried a majority of states, this paradox is not entrenched and disappears with a sufficiently small House size. Again, the crossover House size is so small that adjustments must be made to account for the guaranteed floor.[16]

Table 1 also shows two additional historical elections that are subject to the partial House size effect, because a sufficiently small (in particular, minimum) House size would create an electoral vote tie, as the number of states was even and each candidate carried half of them.

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[14] In Figure 7 of Barthélémy et al. (2014), reflecting their calculations using the Webster apportionment then in use, the crossover House size is literally off the chart.

[15] This conclusion is consistent with Figure 5 in Barthélémy et al. (2014). All this takes the official results at face value, though the vote counts in several states were bitterly disputed and resolved only by a special Electoral Commission.

[16] However, against his actually disunited opposition, Lincoln carried at least 17 states out of 33, including the largest states, and so would have won under any House size. Given rival Democratic ‘fusion’ and ‘straight Douglas’ electors slates on the ballot in New Jersey, it cannot be unambiguously established who ‘carried’ the state, and its electoral votes were in any event split. (In Table 1, New Jersey is credited to the anti-Lincoln opposition.)

6. Conclusion

We conclude by coming full circle and reconsidering Proposition A. While commonsensical and historically accurate, it fails as a theoretical proposition. Indeed, we can state the following.

Proposition 4. For a presidential election to be subject to the House size effect, it neither necessary nor sufficient that:

(1A) one candidate (say A) wins a majority of the Senate electoral votes, *and*
(2A) the other candidate (say B) wins a majority of the House electoral votes.

To show that (1A) and (2A) are not jointly necessary, suppose that candidate A wins a small majority of both Senate and House electoral votes. But it may be that p_B slightly exceeds p_B and that A's advantage with respect House electoral votes only reflect favorable effects of local chaos of the magnitude illustrated in footnote 8; therefore candidate B would win given most other House sizes (and any sufficiently large House size), making the election subject to House size effect. Conversely and to show that they are not jointly sufficient, suppose that p_A only slightly exceeds p_B; then there may be other (small or medium but not very large) House sizes at which candidate B wins more House electoral votes than candidate A but never enough to overcome A's advantage with respect to Senate electoral votes; thus the election is not subject to House size effects.

The last rows in Table 1 show three historical elections that do not meet conditions (1A) and (2A) but which might possibly be subject to House size effects, given that the winning candidate's (call him A) advantage with respect to Senate electoral votes was small and could be overcome at some House size by a small advantage in House electoral votes for candidate B. Since p_A substantially exceeds p_B in all three elections, candidate B could have won a majority of House electoral votes only at House sizes slightly exceeding the minimum (at which only the few largest states have more than one House seat) and if B had carried these largest states. But in fact in each case candidate A carried enough of the largest states to preclude even a partial House size effect.

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Table 1. Selected Historical Presidential Elections

(1) Category of Election	(2) Election	(3) PV Winner	(4) States Carried		(5) Senate EV		(6) House EV (Actual)		(7) Pop. %*		(8) House EV (Perf. Apport.)		(9) Crossover House Size (Perf. App.)
			Dem	Rep	Dem	Rep	Dem	Rep	Dem	Rep	Dem	Rep	
Clearly Subject to House Size Effect	2000**	Dem	21	30	42	60	225	211	51.68	48.32	225.34	210.66	534.64
	1976	Dem	24	27	48	54	249	187	57.37	42.63	250.14	185.86	40.69
	1960[1]	Dem	23	26	46	52	263	168	60.24	38.31	263.24	167.43	29.30
	1916	Dem	30	18	60	36	216	219	49.91	50.09	217.60	218.40	12938.84
	1876**	Dem	17	21	34	42	150	143	51.80	48.20	151.77	141.23	222.48
	1860[2]**	Dem	18	15	36	30	98	139	40.98	59.02	97.11	139.89	33.25
Partial Effect	1880	Rep	19	19	38	38	117	176	40.32	59.64	118.15	174.85	0
	1848	Whig	15	15	30	30	97	133	42.52	57.48	97.79	132.21	0
Possibly Subject (But Not)	1896	Rep	22	23	44	46	131	226	36.22	63.78	129.29	227.71	–
	1888***	Dem	18	20	36	40	132	193	41.04	58.96	133.39	191.61	–
	1884	Dem	20	18	40	36	179	146	52.51	47.49	179.86	145.14	–

* Percent of apportionment population in states carried

** Referendum Paradox with actual House size (not 'entrenched')

*** Referendum Paradox with any House size ('entrenched')

[1] Kennedy is credited with carrying Alabama and winning all of its electoral votes; an unpledged elector slate carried Mississippi (8 electoral votes)

[2] Counterfactual two-candidate election: Lincoln vs. united opposition

Source: Popular and electoral vote data are available from many sources; the most authoritative may be *Guide to U.S. Elections*, 5th ed., CQ Press, 2005. Apportionment populations are from Balinski and Young (2001), Appendix B; their figures come from the Census Bureau and other sources noted on p. 157. When a state was admitted since the last decennial census and apportionment, its territorial population in the previous census is used, with the exception of Minnesota and Oregon in 1860, for which their (state) population in the 1860 census were used. (Minnesota and Oregon were admitted as states in 1858 and 1859; both were virtually unpopulated in 1850.) Territorial populations are taken from *http://en.wikipedia. org/wiki/ List_of_U.S._states_by_historical_population*, which in turn cites the U.S. Census.