Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 168 (2011) 89–101
FUZZY sets and systems
www.elsevier.com/locate/fss

The uncovered set and indifference in spatial models: A fuzzy set approach

John N. Mordeson[a], Terry D. Clark[b,*], Nicholas R. Miller[c], Peter C. Casey[d], Michael B. Gibilisco[b]

[a] *Department of Mathematics, Creighton University, United States*
[b] *Department of Political Science, Creighton University, Omaha, NE 68123, USA*
[c] *Department of Political Science, University of Maryland Baltimore County, United States*
[d] *Department of Political Science, Washington University in St. Louis, United States*

Received 24 June 2009; received in revised form 25 October 2010; accepted 26 October 2010
Available online 9 November 2010

Abstract

The uncovered set was developed in order to predict outcomes when spatial models result in an empty core. In contrast to conventional approaches, fuzzy spatial models induce a substantial degree of individual and collective indifference over alternatives. Hence, existing definitions of the covering relationship return differing results. We develop a definition for a fuzzy covering relation. Our definition results in an uncovered set that is, in most cases, contained within the Pareto set. We conclude by characterizing the exceptions.

Keywords: Social sciences; Fuzzy preferences; Spatial models; Cycling; Uncovered set

1. Introduction

There are a number of incongruences between the predictions made by conventional spatial models in political science and empirical reality. The most well-known incongruence results from pervasive cycling of majority rule in a space of two or more dimensions. In fact, however, there is little empirical evidence of cycling in political life [1]. An approach to resolving the majority cycling problem, that has been long known, but remains largely under-developed, is the introduction of thick indifference in individual preferences in models [2]. Several studies have found that the probability of a majority rule maximal set increases when actors are indifferent over regions of the policy space [3–5]. Much of this work makes use of the epsilon-core. Until the distance between two alternatives exceeds some arbitrary distance ε, actors are indifferent in a choice between them [6,7]. Actors are essentially indifferent to alternatives within radius ε from one another, in a region defined as the ε-core. While this is an interesting approach, its utility is undermined by the difficulty that conventional mathematical tools encounter when individual preferences are irregularly shaped in

* Corresponding author. Tel.: +1 402 280 4712; fax: +1 402 280 4731.
E-mail address: tclark@creighton.edu (T.D. Clark).

0165-0114/$ - see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.fss.2010.10.016

the policy space, which is highly likely when issue dimensions are non-separable. Thus, empirical testing of models of indifference attempting to make use of the ε-core concept becomes problematic.

A more promising approach to incorporating indifference in individual preferences in spatial models is offered by fuzzy set theory and is explicated in [8]. The approach not only permits the modeling of a substantial degree of indifference, it also thickens the line that defines a player's wincircle. Furthermore, in contrast to the ε-core concept, it models uniform indifference over a fixed and discrete region of the policy space. However, it is shown in [9] that while the approach to modeling fuzzy individual preferences in [8] significantly enhances the likelihood of a stable maximal rule outcome, it does not guarantee its existence in all cases. This raises the issue of how to arrive at a prediction set when spatial models do not result in a maximal set?

One strand of the public choice literature argues that, in the absence of a majority rule maximal set, political actors should choose an alternative in the uncovered set, the set of alternatives that sophisticated voters would reach by some amendment agenda [10–12]. The specific alternative chosen depends on the amendment order. This paper develops an uncovered set induced by players with fuzzy preferences. We follow the lead in [8] and adopt discrete fuzzy numbers to represent the preferences of political actors. We demonstrate that with one relatively trivial exception, the uncovered set comprises a subset of the Pareto set (Theorem 4.11).

As shown in [9], the fuzzy approach to modeling individual preferences in [8] relies on a general theory in which a region of interest is mapped to a region with a suitable and natural partial ordering. The partial ordering plays a significant role in the determination of the results. The mapping involved then transfers the results faithfully back to the original region of interest. The homomorphism permits a straightforward calculation of majority rule maximal sets, Pareto sets, and the uncovered set, and thereby enhances empirical testing of the models.

We are not the first to offer a fuzzy approach to spatial modeling. As laid out in [13] a great deal of effort has gone into considering the challenge. However, the majority of efforts have hithertofore focused on preference relations [14–21]. [1] We gratefully acknowledge the contribution that this work has made to our own thinking on the subject. Nonetheless, we note that data in the social sciences do not easily lend themselves to measuring preference relations. If models are to be tested, they must take individual preference as their starting point. [2] We are interested in developing an approach to spatial modeling that is sensitive to the demands of the NSF-sponsored empirical implications of theoretical models (EITM) movement [22] that models be testable. This paper is part of a long-term research project to produce such models. For that reason, our approach is a new one that follows the lead of [23] in modeling fuzzy individual preferences and builds upon work in [8]. It represents the next step following work in [9], which fully characterizes the conditions under which the majority rule maximal set is empty.

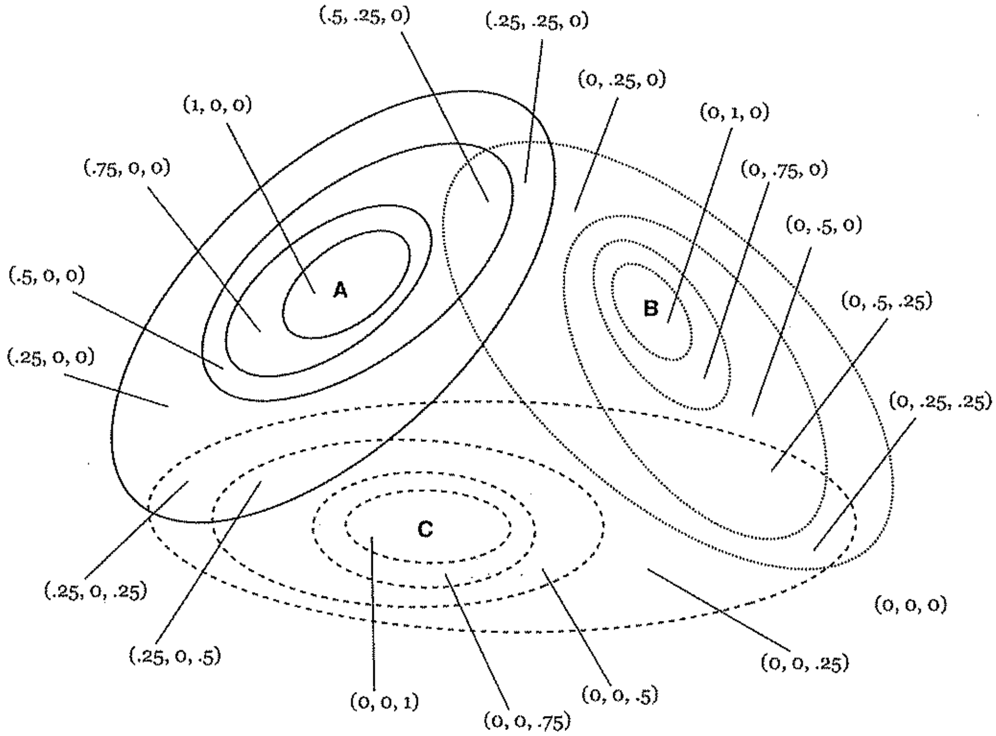
2. Modeling fuzzy individual preferences

In [8] it is argued that fuzzy set theory permits the modeling of a substantial degree of indifference in political actors' preferences over policy options. Each element x in the universe of policy alternatives X over which political players are making choices are assigned a value on the interval [0,1]. The assigned value indicates the degree to which a political actor considers the given alternative to be an ideal policy. Formally, each element x has a degree of set inclusion, $\tilde{F}(x)$, in the set of ideal preferences that is specified by a function $\tilde{F} : X \to [0, 1]$ [24]. For α in [0,1], the sets $\tilde{F}^{\alpha} = \{x \in X \mid \tilde{F}(x) \geq \alpha\}$ are referred to as alpha-levels (α-levels).

Individual indifference is captured by the α-level concept. A political player is indifferent between all alternatives at the same α-level, essentially treating them as equivalents. Alpha-levels may take any value, which permits modelers to distinguish between actors who are more or less discerning in their preferences. The conventional approach locks one into modeling players who have the capacity to make infinitesimally small distinctions between alternatives. While a fuzzy approach can accommodate such an assumption by assigning α-levels along the continuum from 0 to 1, individual preferences over policies are likely to be substantially less discriminating. Following the lead in [14], we model such coarse-grained distinctions using a finite set of discrete α-levels $T = \{0, .25, .5, .75, 1\}$. This Likert-like scale has an intuitive appeal to it. The set of ideal policy preferences for an actor are assigned an α-level of 1 (full membership in the set of ideal policies). All remaining policies are scored on the degree to which they come close to being ideal. Those considered almost ideal are assigned .75 (three-fourths membership in the set of ideal policies), those considered

[1] Preference relations are understood formally as $\rho : X^2 \to [0, 1]$.

[2] Individual preferences are understood formally as $\sigma : X \to [0, 1]$.



Three-tuple	Winset
(1, 0, 0)	(0, .5, .25), (0, .25, .25)
(.75, 0, 0)	(0, .5, .25), (0, .25, .25)
(.5, .25, 0)	(0, .5, .25)
(.5, 0, 0)	(0, .5, .25), (0, .25, .25)
(.25, .25 0)	(0, .5, .25)
(.25, 0, .5)	(.5, .25, 0)
(.25, 0, .25)	(.5, .25, 0)
(.25, 0, 0)	(0, .5, .25), (0, .25, .25)
(0, 1, 0)	(.25, 0, .5), (.25, 0, .25)
(0, .75, 0)	(.25, 0, .5), (.25, 0, .25)
(0, .5, .25)	(.25, 0, .5)
(0, .5, 0)	(.25, 0, .5), (.25, 0, .25)
(0, .25, .25)	(.25, 0, .5)
(0, .25, 0)	(.25, 0, .5), (.25, 0, .25)
(0, 0, 1)	(.5, .25, 0), (.25, .25, 0)
(0, 0, .75)	(.5, .25, 0), (.25, .25, 0)
(0, 0, .5)	(.5, .25, 0), (.25, .25, 0)
(0, 0, .25)	(.5, .25, 0), (.25, .25, 0)
(0, 0, 0)	(.5, .25, 0), (.25, .25, 0), (.25, 0, .5), (.25, 0, .25), (0, .5, .25), (0, .25, .25)

Fig. 1. An empty maximal set in a fuzzy spatial model.

"neither ideal nor not ideal" are scored .50 (one-half membership in the set of ideal policies), those "less ideal than ideal" are scored .25 (one-quarter membership in the set of ideal policies), and those that are "not ideal" to any degree 0 (no membership in the set of ideal policies).

Since all alternatives at the same α-level are equally preferred to one another, a player is indifferent to all alternatives at a given α-level. Thus, fuzzy spatial models map individual preferences as bounded regions (defined by each α-level), within which a political player cannot differentiate among policy positions.

Consider Fig. 1, which maps the preferences of three players $N = \{\tilde{A}, \tilde{B}, \tilde{C}\}$ at $\alpha = 1, .75, .50$, and .25 in two-dimensional space. The inner-most regions represent $\alpha = 1$. The intersection of the α-levels for the three players (three-tuples) are noted in parentheses, $(\tilde{A}(x), \tilde{B}(x), \tilde{C}(x))$. If a maximal set exists under majority rule, it falls in one of the intersections of a majority of players' α-levels.

Majority rule is the complete binary relation R such that xRy means that the number of voters who prefer x to y is at least as great as the number who prefer y to x. If xRy and yRx, we say "x ties y." P is the asymmetric component of R. That is, xPy implies xRy and $\sim yRx$ and that the number of voters who prefer x to y is greater than the number who prefer y to x. Formally, we define majority rule as

$$xPy \equiv |\{i \in N | xP_i y\}| > \frac{n}{2}, \tag{1}$$

where N is the set of all players and n is the number of players. If x defeats y under majority rule, then $xPy \equiv |\{i \in N | xP_i y\}| > n/2 > |\{j \in N | yP_j x\}|$, in which case we say "$x$ beats y" under majority rule.

The maximal set is the set of alternatives that are maximal with respect to R or, put otherwise, the set of unbeaten alternatives. Formally, the maximal set is defined as

$$M(R, X) = \{x \in X | \forall y \in X, \sim yPx\}, \tag{2}$$

where X is the set of alternatives. Let N denote the set of players and let n denote the number of players. Let $\mathcal{R}$ denote the set of all binary relations on X that are reflexive and complete. Let $\mathcal{R}^n$ denote the set of all n-tuples $\rho = (R_1, \ldots, R_n)$, where R_i belongs to $\mathcal{R}$, $i = 1, \ldots, n$. Let f be a majority aggregation rule on $\mathcal{R}^n$. Following convention, we suppress the notation $f(\rho)$ and simply write R. Then P denotes the strict preference relation associated with $f(\rho)$.

The maximal set may be empty because majority rule may cycle, e.g., xPy, yPz, and zPx. Given a profile of Euclidean preferences over a two-dimensional (or space) of alternatives, majority rule cycling is almost always pervasive. If the number of voters is odd, there is at most one core alternative, but it exists only under conditions of "generalized Plott [25] symmetry" [26]. If the number of voters is even, at most one core alternative exists and only under similarly stringent symmetry conditions.[3] In the absence of such symmetry, majority rule cycling encompasses the entire alternative space [28,29].

There is no majority rule maximal set in the situation defined by the sets of preferences depicted in Fig. 1. The set of options that are majority preferred (the winset) to each numbered alternative are noted. All alternatives are majority preferred by at least one other alternative.

The potential for cycling in spatial models of fuzzy preferences leaves open the question of how to predict outcomes in the absence of a majority rule maximal set. This paper develops an uncovered set as an alternative prediction set when players' preferences are fuzzy and no majority rule maximal set exists. In what follows, we demonstrate that with one exception the uncovered set is a subset of the Pareto set (Theorem 4.11). As it turns out, the exception is trivial, since it requires that no alternative is majority preferred to any other.

3. A definition of covering

The uncovered set is the set of sophisticated voting outcomes in an amendment agenda. Miller [10] originally examined the covering relation and the uncovered set in majority preference tournaments (amendment agendas) that result when an odd number of voters have strong preferences over discrete alternatives. Formally, the uncovered set is

[3] With just four voters and two dimensions, a unique core alternative always exists. When there are four voters, majority rule is equivalent to three-quarters majority rule; and Greenberg's Theorem [27] tells us that an unbeaten alternative always exists when a decision rule is more demanding than $M/(M+1)$, where M is the dimensionality of the space.

defined as

$$UC(X) = \{x \in X \mid \sim yCx \; \forall y \in X\}, \tag{3}$$

where C is a covering relation. Miller [10,30] assumed that players would exercise strict preference over all alternatives. He defined the covering relation as

$$xCy \equiv W^{-1}(y) \subset W^{-1}(x), \tag{4}$$

where $W^{-1}(x)$ is the set of all alternatives to which x is strictly preferred by a majority. $W^{-1}(x)$ is everything that x beats. $W^{-1}(x)$ is called the inverse winset of x. Formally, $W^{-1}(x) = \{y \in X \mid xPy\}$. In other words, x covers y (xCy) if and only if x is at least as good as y and every point strictly beaten by y is also strictly beaten by x and there exists w in X such that x strictly beats w and y does not strictly beat w.

A number of alternative definitions of the covering relations C have subsequently been proposed. In the conventional approach to spatial modeling, indifference among an odd number of players is "thin." That is, $a \neq b \Rightarrow aPb$ or bPa, where P denotes the strict preference relation. (In conventional Euclidean spatial models, indifference is limited to the indifference curves that describe individual preference.) Under conditions of thin indifference, all of the definitions of the covering relation in the literature are equivalent in a spatial context. However, Penn [31] and Miller [32] have noted that in any context that allows ties, these equivalences break down.

In the case of spatial models of fuzzy individual preferences, both individual indifference and collective indifference (tie sets) are "thick": $a \neq b \Rightarrow aRb$ or bRa, where R denotes the weak preference relation. As a consequence, the equivalence relations among the various proposed definitions of the covering relation do not hold; and the definitions of the covering relation return differing results [31,33].

Collective indifference is not uncommon when individual players possess fuzzy preferences over alternatives, even when the number of players is odd. Consider the three-player case. The three-tuple representing the alternative lying at the intersection of the preferences (α-levels) of three players at (.75,.5,.25) is not majority preferred to that at (.5,.75,.25). Player one prefers the first alternative, and player two prefers the second alternative, but player three is indifferent in a choice between them.

Collective indifference leaves open the possibility that a given definition of covering will result in an uncovered set that contains alternatives that are not in the set of sophisticated voting outcomes in an amendment agenda. We need a definition of covering that identifies an uncovered set containing only such alternatives under conditions of thick indifference induced by fuzzy individual preferences. This is an important characteristic of the uncovered set, without which its claim as a solution in the absence of a majority rule maximal set is rendered void. In what follows, we identify such a definition.

We are guided in our task by Miller [10] and Shepsle and Weingast [11] who formalize a process for determining the sophisticated outcome in an amendment agenda. Their process focuses on levels in the voting order rather than the decision nodes in the voting tree representing an amendment agenda. Formally, a voting tree is "an amendment agenda if there exists an ordering $\gamma : \{1, \ldots, r\} \rightarrow X$ such that the majority voting sequence is $(\gamma(1), \gamma(2), \ldots, \gamma(r))$, where $\gamma(1)$ is first voted against $\gamma(2)$, the winner against $\gamma(3)$, etc." [34, p. 132].

We demonstrate Shepsle and Weingast's approach as follows. Suppose that we have a cycle set: (.25,.25,.5)P(.75,0,.25) P(.5,1,0)P(.25,.75,1)P(1,.5,.75)P(.25,.25,.5). Let the agenda order A be

$$A = \{(.25, .25, .5), (1, .5, .75), (.5, 1, 0), (.25, .75, 1), (.75, 0, .25)\}.$$

Any alternative chosen by sophisticated players must be preferred by a majority to the final alternative in the voting order. Those that fail to do so, cannot be the sophisticated outcome. Furthermore, the sophisticated outcome must be majority preferred to all alternatives that are majority preferred at any level γ of the game. Following the Shepsle and Weingast procedure, the step-by-step results are

$\gamma(5) = (.75, 0, .25)$
$\gamma(4) = (.25, .75, 1)$, (.25,.75,1)$P$(.75,0,.25)
$\gamma(3) = (.25, .75, 1)$, (.5,.1,0)$P$(.25,75,1) but (.75,0,.25)P(.5,1,0)
$\gamma(2) = (.25, .75, 1)$, (1,.5,.75)$P$(.75,0,.25) but (.25,.75,1)P(1,.5,.75)
$\gamma(1) = (.25, .75, 1)$, (.25,.25,.5)$P$(.75,0,.25) but (.25,.75,1)P(.25,.25,.5)

By reflexivity, the final alternative is at least as good as itself. Hence, in the construction of the backward induction process that is common to determining the sophisticated winner in a voting game, (.75,0,.25) is trivially the winner at the level $\gamma(5)$. At the next level, (.25,.75,1) is majority preferred to (.75,0,.25). Hence, it is the alternative at level $\gamma(4)$. While (.5,1,0) defeats (.25,.75,1) at level $\gamma(4)$, it is defeated by (.75,0,.25) at level $\gamma(5)$. Thus, (.25,.75,1) carries over as the winner at level $\alpha(3)$. By the same reasoning, (.25,.75,1) carries over as the winner at levels $\gamma(2)$ and $\gamma(1)$. Since it is the winner at level $\gamma(1)$, it is the sophisticated majority winner.

While this particular voting order appears to suggest that Shepsle and Weingast's [11] method can be used with fuzzy preferences, the problem induced by indifference comes into full view if we consider the voting agenda

$$A = \{(.75, .25, 0), (.25, 0, .75), (0, .5, .75), (.75, 0, .5), (.75, .5, 0)\}.$$

The results through the first three levels are:

$\gamma(5) = (.75, .5, 0)$
$\gamma(4) =$ outcome uncertain, $(.75,.5,0)I(.75,0,.5)$
$\gamma(3) =$ outcome uncertain, $(0,.5,.75,)P(.75,.0,.5)$ but $(0,.5,.75)I(.75,.5,0)$

In this case, (.75,.5,0) is trivially majority preferred at level $\gamma(5)$, but it is tied (it is indifferent, as noted by I) with (.75,0,.5) at level $\gamma(4)$. Furthermore, while (0,.5,.75) defeats (.75,0,.5) by a majority, it is indifferent to (.75,.5,0).

We are faced with two issues: (1) how to proceed when an alternative defeats at least one alternative majority preferred at previous levels but ties all others and (2) how to proceed when an alternative ties all alternatives majority preferred at previous levels? Issue (1) is the easiest of the two challenges to deal with. An alternative that defeats at least one alternative more than a successor in the voting order is superior to the successor. Therefore, we should designate it the winner at the given level. However, in the case of issue (2), since neither alternative is superior in this sense, we are compelled to accept having to list both at the given level. This forces us to compare subsequent predecessors with both alternatives. The guiding principle is that the sophisticated outcome is one that majority defeats one or more majority preferred successors and ties all others.

Using this procedure, we get

$\gamma(5) = (.75, .5, 0)$	
$\gamma(4) = (.75, .5, 0)\&(.75, 0, .5)$,	$(.75,.5,0)I(.75,0,.5)$
$\gamma(3) = (0, .5, .75)$,	$(0,.5,.75,)P(.75,0,.5)$ and $(0,.5,.75)I(.75,.5,0)$
$\gamma(2) = (0, .5, .75)$,	$(.25,0,.75)I(0,.5,.75),(.25,0,.75)I(.75,0,.5)$, but $(.75,.5,0)P(.25,0,.75)$
$\gamma(1) = (0, .5, .75)$,	$(.75,.25,0)I(.75,.5,0),(.75,.25,0)I(.75,0,.5)$, but $(0,.5,.75)P(.75,.25,0)$

The alternative represented by the preference three-tuple (0,.5,.75) is the sophisticated outcome given this amendment agenda, and it is an element in the uncovered set.

The procedure that we followed when determining the sophisticated outcomes under thick indifference meets two criteria. First, for an alternative x to be a sophisticated outcome at any level γ, it must be at least as good as all previous sophisticated outcomes. Second, for an alternative x to eliminate a previous sophisticated outcome y, then $W^{-1}(y) \subset W^{-1}(x)$. These criteria are captured in the following definition of the covering relation:

$$xCy \Leftrightarrow xRy\&W^{-1}(y) \subset W^{-1}(x). \tag{5}$$

In other words, x is at least tied with y, x beats everything that y beats, and x beats something (perhaps y itself) that y fails to beat. Bordes [35] labels the uncovered set that results from this definition the FD set. It differs from any of the three definitions considered by Penn [31], all of which imply that x is strictly preferred to y, xPy. Henceforth, we use this definition of covering and call it C.

In what follows, we give formal consideration to C. Our major theorem, Theorem 4.11, demonstrates that with one trivial exception, any element in $UC(X)$ is in the Pareto set. Formally, the Pareto set is defined as in [34, p. 7]. $\forall \overline{p} \in \mathbb{R}^n$,

$$PS_N(\overline{p}) = \{x \in X | \forall y \neq x,\ P(y, x; \overline{p}) \neq \emptyset \Rightarrow P(x, y; \overline{p}) \neq \emptyset\}, \tag{6}$$

where $P(x, y; \overline{p}) \equiv \{i \in N | x P_i y\}$.

4. The uncovered set

Let N denote the set of players and X denote the set of alternatives. We assume that X is a subset of a universe U of interest, where U is arbitrary through Proposition 4.7. Let $\mathcal{R}$ denote the set of all binary relations on X that are reflexive and complete. Let $\mathcal{R}^n = \{\rho | \rho = (R1, \ldots, R_n), R_i \in \mathcal{R}, i = 1, \ldots, n\}$, where $|N| = n$. Let $\preceq$ be any partial order on U. $\forall x, y \in U, x \prec y$ iff $x \preceq y$ and $x \neq y$. Suppose that $\preceq$ satisfies the following properties:

(1) $\forall x, y \in U, x \preceq y$ implies $\forall i \in N, yR_i x$;
(2) $\forall x, y, z \in U, \forall i \in N, x \preceq y$ and $xR_i z$ implies $yR_i z$;
(3) $\forall x, y, z \in U, \forall i \in N, x \preceq y$ and $xP_i z$ implies $yP_i z$;
(4) $\forall x, y \in U, x \prec y$ implies $\exists i \in N$ such that $yP_i x$;
(5) $\forall x, y, z \in U, \forall i \in N, x \preceq y$ and $zR_i y$ implies $zR_i x$;
(6) $\forall x, y \in U$, x and y incomparable under $\preceq$ implies $\exists i \in N$ such that $xP_i y$ implies $\exists j \in N$ such that $yP_j x$.

Definition 4.1. Define $\langle\,\rangle : P(U) \to P(U)$ by $\forall S \in P(U), \langle S\rangle = \{x \in U | \exists s \in S, x \preceq s\}$.

Proposition 4.2. *Let $\langle\,\rangle : P(U) \to P(U)$ be defined as above. Then the following conditions hold:*

1. $\forall S \in P(U), S \subseteq \langle S\rangle$;
2. $\forall S_1, S_2 \in P(U), S_1 \subseteq S_2$ *implies* $\langle S_1\rangle \subseteq \langle S_2\rangle$;
3. $\forall S \in P(U), \langle S\rangle = \langle\langle S\rangle\rangle$;
4. $\forall S \in P(U), \langle S\rangle = \bigcup_{s\in S}\langle\{s\}\rangle$;
5. $\forall S \in P(U), \forall x, y \in X, x \in \langle S \cup \{y\}\rangle$ *and* $x \notin \langle S\rangle$ *implies* $x \in \langle\{y\}\rangle$.

Proof. (1) Let $s \in S$. Then $s \preceq s$ and so $s \in \langle S\rangle$. Thus $S \subseteq \langle S\rangle$.

(2) Let $x \in \langle S_1\rangle$. Then there exists $s \in S_1$ such that $x \preceq s$. Since $s \in S_2, x \in \langle S_2\rangle$.

(3) By (1), $\langle S\rangle \subseteq \langle\langle S\rangle\rangle$. Let $x \in \langle\langle S\rangle\rangle$. Then there exists $y \in \langle S\rangle$ such that $x \preceq y$. There exists $s \in S$ such that $y \preceq s$. Since $\preceq$ is transitive, $x \preceq s$. Thus $x \in \langle S\rangle$. Hence $\langle\langle S\rangle\rangle \subseteq \langle S\rangle$.

(4) For all $s \in S$, $\langle\{s\}\rangle \subseteq \langle S\rangle$ by (2). Thus $\bigcup_{s\in S}\langle\{s\}\rangle \subseteq \langle S\rangle$. Let $x \in \langle S\rangle$. Then there exists $s \in S$ such that $x \preceq s$. Thus $x \in \langle\{s\}\rangle$ and so $x \in \bigcup_{s\in S}\langle\{s\}\rangle$. Hence $\langle S\rangle \subseteq \bigcup_{s\in S}\langle\{s\}\rangle$.

(5) Suppose $x \in \langle S \cup \{y\}\rangle$ and $x \notin \langle S\rangle$. Then there does not exist $s \in S$ such that $x \preceq s$. Hence $x \preceq y$. Thus $x \in \langle\{y\}\rangle$. □

Definition 4.3. Let $\overline{\rho} \in \mathcal{R}^n$. Define the binary relation R on X by $\forall x, y \in X, (x, y) \in R$ if and only if $|\{i \in N | xR_i y\}| \geq n/2$. Define $P \subseteq X \times X$ by $\forall x, y \in X, (x, y) \in P$ if and only if $(x, y) \in R$ and $(y, x) \notin R$. Let $R(x, y; \overline{\rho}) = \{i \in N | xR_i y\}$ and $P(x, y; \overline{\rho}) = \{i \in N | xP_i y\}$.

Definition 4.3 gives the relationship between $\overline{\rho}$ and R in what follows.

Definition 4.4. Let $M_R = \{x \in X | \nexists y \in X, x \prec y\}$.

Proposition 4.5. *Let $\overline{\rho} \in \mathcal{R}^n$ and let R be defined as in Definition* 4.3. *Then $M_R = PS_N(\overline{\rho})$.*

Proof. Suppose $x \in M_R$. Let $y \in X$. Suppose $\exists i \in N$ such that $yP_i x$. Now there does not exist $y \in X$ such that $x \prec y$. Thus $\forall y \in X$, either $y \preceq x$ or x and y are not comparable. Since $yP_i x$, $y \preceq x$ is impossible else $xR_i y$ $\forall i \in N$ by (1). Hence x and y are incomparable under $\preceq$. Thus $\exists j \in N$ such that $xP_j y$ by (6). Hence $x \in PS_N(\overline{\rho})$. Thus $M_R \subseteq PS_N(\overline{\rho})$.

Suppose $x \in PS_N(\overline{\rho})$. Suppose there exists $y \in X$ such that $x \prec y$. Then $\exists i \in N$ such that $yP_i x$ by (4). Since $x \in PS_N(\overline{\rho})$, there exists $j \in N$ such that $xP_j y$. Thus $x \prec y$ is impossible. Hence $x \in M_R$. Therefore $PS_N(\overline{\rho}) \subseteq M_R$. □

Corollary 4.6. *Let $x \in X$.*

(1) *Suppose $\forall y \in X, x \preceq y$ implies $x = y$. Then $x \in PS_N(\overline{\rho})$.*
(2) *If $x \notin PS_N(\overline{\rho})$, then there exists $y \in PS_N(\overline{\rho})$ such that $x \prec y$.*

Proof. (1) Clearly $x \in M_R$, but $M_R = PS_N(\overline{\rho})$.

(2) Since $x \notin PS_N(\overline{\rho})$, $x \notin M_R$. Thus there exists $y \in X$ such that $x \prec y$. Let y be the largest such element. Then $y \in M_R = PS_N(\overline{\rho})$. □

Proposition 4.7. $\langle X \rangle = \langle PS_N(\overline{\rho}) \rangle$.

Proof. Clearly, $PS_N(\overline{\rho}) \subseteq X$. Thus $\langle PS_N(\overline{\rho}) \rangle \subseteq \langle X \rangle$. Let $x \in X$. If $x \notin \langle PS_N(\overline{\rho}) \rangle$, then $x \notin PS_N(\overline{\rho})$ and so by (2) of Corollary 4.6, there exists $y \in PS_N(\overline{\rho})$ such that $x \prec y$. Thus $x \in \langle \{y\} \rangle \subseteq \langle PS_N(\overline{\rho}) \rangle$. Hence $X \subseteq \langle PS_N(\overline{\rho}) \rangle$ and so $\langle X \rangle \subseteq \langle PS_N(\overline{\rho}) \rangle$. □

In the remainder of the paper, we assume $X = \langle X \rangle \subseteq T^n$, where $T = \{0, .25, .5, .75, 1\}$. Let $J_{n/2} = \{(a_1, \ldots, a_n) \in T^n | a_i \in \{0, 1\}, i = 1, \ldots, n, |\{i | a_i = 0\}| \geq n/2$ and $\exists j \in N$ such that $a_j = 1\}$. Let $\overline{1_i} = (a_1, \ldots, a_n) \in T^n$ be such that $a_i = 1$ and $a_j = 0$ for $j = 1, \ldots, n$; $j \neq i$. In the remainder of the paper, we assume $\{\overline{1_i} | i = 1, \ldots, n\} \subseteq X$. This must be the situation in our application. Hence, $J_{n/2} \cap U \neq \emptyset$. In the following result we use the notation [[]] to denote the greatest integer function.

Proposition 4.8. *Let $x, y \in X$. If xCy, then either $x \succ y$ or $y \in \langle J_{n/2} \rangle$.*

Proof. Suppose $x \preceq y$. Then clearly $\nexists w \in X$ such that xPw and not yPw, a contradiction since xCy. Thus either $x \succ y$ or x and y are not comparable with respect to $\preceq$. Suppose x and y are not comparable with respect to $\preceq$, where $x = (x_i, \ldots, x_n)$ and $y = (y_i, \ldots, y_n)$. Then $\exists i, j \in N$ such that $x_i > y_i$ and $x_j < y_j$. Since xCy, xRy. Thus not yPx. Hence strictly fewer than $[[n/2]] + 1$ of the y_i are strictly greater than the corresponding x_i. There is no loss in generality in assuming $y_1 \leq x_1, \ldots, y_r \leq x_r$ and $y_{r+1} > x_{r+1}, \ldots, y_n > x_n$, where $n - r < [[n/2]] + 1$. Suppose $y_1 = \cdots = y_{s-1} = 0$ for $s \geq 1$. (The case $s = 1$ says no $y_i = 0$.) We show $s - 1 \geq n/2$. Assume $s - 1 < n/2$. Then $n - s + 1 > n/2$. Let t be such that $r - t - s + 1 + n - r = [[n/2]] + 1$. Now let $y'_s, \ldots, y'_{r-t}$ be such that $y_s > y'_s \geq 0, \ldots, y_{r-t} > y'_{r-t} \geq 0$. Let

$$z = (x_1, \ldots, x_{s-1}, \overbrace{y'_s, \ldots\ldots, y'_{r-t}}^{r-t-s+1}, \overbrace{x_{r-t+1} \ldots, x_r}^{t}, x_{r+1}, \ldots, x_n).$$

Now $n - t - s + 1 = [[n/2]] + 1$. Thus

$$t + s - 1 = \begin{cases} \left[\left[\dfrac{n}{2}\right]\right] - 1 & \text{if } n \text{ is even,} \\ \left[\left[\dfrac{n}{2}\right]\right] & \text{if } n \text{ is odd.} \end{cases}$$

Since $y'_s < y_s \leq x_s, \ldots, y'_{r-t} < y_{r-t} \leq x_{r-t}$ and $X = \langle X \rangle$, $z \in X$. Now $t + s - 1$ is the number of components x_i is strictly greater than the corresponding components of z. Thus not xPz. However, $r - t - s + 1 + n - r = [[n/2]] + 1$ (see above) is the number of components of y that are strictly greater than the corresponding components of z. Thus yPz, contradicting the hypothesis that xCy. Thus $s - 1 \geq n/2$. Hence $y \in \langle J_{n/2} \rangle$. □

Proposition 4.9. *Let $x, y \in X$. If $x \succ y$, then either ($\exists w \in X$ such that xPw and not yPw) or $x \in \langle J_{n/2} \rangle$.*

Proof. Suppose x and y differ in $[[n/2]] + 1$ or more components, where [[]] denotes the greatest integer function. Then let $w = y$. Suppose x and y differ in fewer than $[[n/2]] + 1$ components. There is no loss in generality in assuming that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, where $x_1 = y_1, \ldots, x_r = y_r, x_{r+1} > y_{r+1}, \ldots, x_n > y_n$, and $n - r \leq [[n/2]]$. Thus

$$r \geq n - \left[\left[\frac{n}{2}\right]\right] = \begin{cases} \left[\left[\dfrac{n}{2}\right]\right] & \text{if } n \text{ is even,} \\ \left[\left[\dfrac{n}{2}\right]\right] + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Suppose $x \notin \langle J_{n/2} \rangle$. Then fewer than $n/2$ of the $x_i = 0$, say $x_1 = \cdots = x_s = 0$, where $s < n/2$. Thus $s < r$. Let t be such that $t - s + n - r = [[n/2]] + 1$. Let

$$w = (x_1, \ldots, x_s, \overbrace{w_{s+1}, \ldots, w_t}^{t-s}, \overbrace{y_{t+1}, \ldots, y_r}^{r-t}, \overbrace{y_{r+1}, \ldots, y_n}^{n-r}),$$

where $x_{s+1} > w_{s+1} \geq 0, \ldots, x_t > w_t \geq 0$. Since $x_{s+1} > w_{s+1}, \ldots, x_t > w_t$ and $x_i \geq y_i, \ldots, x_n \geq y_n$ and $X = \langle X \rangle$, $w \in X$. Now $t - s + n - r$ of the components of x are strictly greater than the corresponding components of w. Thus xPw. Now $t - s$ components of y are greater than the corresponding components of w and $t - s \leq [[n/2]]$. Hence not yPw. □

Corollary 4.10. *Let $x, y \in X$. If $x \succ y$, then either xCy or $x \in \langle J_{n/2} \rangle$.*

Proof. The proof follows from condition (3), Definition 4.3, and Proposition 4.9. □

Theorem 4.11. *Let $x \in X$. Suppose x is C-uncovered. Then either $x \in PS_N(\overline{\rho})$ or $x \in \langle J_{n/2} \rangle$.*

Proof. Suppose $x \notin PS_N(\overline{\rho})$ and $x \notin \langle J_{n/2} \rangle$. Since $x \notin PS_N(\overline{\rho})$, there exists $y \in X$ with $y \succ x$ by (2) of Corollary 4.6. By the previous corollary, either yCx or $y \in \langle J_{n/2} \rangle$. But, since $x \notin \langle J_{n/2} \rangle$ and $y \succ x$, then $y \notin \langle J_{n/2} \rangle$. On the other hand, yCx contradicts that x is C-uncovered. □

In other words, all elements in the uncovered set are either in the Pareto set or they are in a special set of alternatives, $\langle J_{n/2} \rangle$. At first blush, the latter possibility appears to put a severe constraint on the ability of the definition of the covering relationship developed in this paper to result in a reasonably constructed uncovered set. Closer scrutiny, reveals the exception to be relatively trivial. $J_{n/2}$ comprises all those alternatives that less than half of the players consider to be perfectly in the set of ideal points ($\alpha = 1$) and all remaining players consider entirely not in the set of ideal points ($\alpha = 0$). $\langle J_{n/2} \rangle$ is the set of all alternatives that descend directly from those in $J_{n/2}$. By definition, none of the alternatives in either $J_{n/2}$ or $\langle J_{n/2} \rangle$ can defeat any alternative by majority vote. Hence, they cannot cover any other alternative. Therefore, if $X \subseteq \langle J_{n/2} \rangle$, then every element of X is C-uncovered. Moreover, if there is even one alternative for which a majority of players express the slightest degree of preference (that is, a majority prefer the alternative at $\alpha > 0$), then every alternative in $\langle J_{n/2} \rangle$ is covered by that alternative and by definition cannot be in the uncovered set. While the alternative may tie elements in $\langle J_{n/2} \rangle$, it can defeat all alternatives lying in the region outside of the support for all players' preferences (that is, all players' preferences for these alternatives are $\alpha = 0$). Hence, it covers elements in $\langle J_{n/2} \rangle$. Thus, elements in $\langle J_{n/2} \rangle$ are uncovered if and only if $\langle J_{n/2} \rangle$ are the only elements in X. That is, no alternatives are supported by a majority at any α-level.

The following corollary summarizes the previous remarks.

Corollary 4.12. *If x is C-uncovered and $x \in \langle J_{n/2} \rangle$, then every element of X is uncovered.*

5. The implication of missing n-tuples

We start with an example showing how n-tuples may be missing. Let $\mathbb{R}_+$ denote the set of nonnegative real numbers and $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}_+$. Let $n = 3$ in this section, and let the players' preferences be denoted by $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$. Let $x \in \mathbb{R}_+^2$. Suppose $x \in \tilde{A}^r \cap \tilde{B}^s \cap \tilde{C}^t$ and $x \notin \tilde{A}^{r'} \cap \tilde{B}^{s'} \cap \tilde{C}^{t'}$ for $r' \geq r, s' \geq s, t' \geq t$, where one of the inequalities is strict. Let $f^* : \mathbb{R}_+^2 \to T^3$ be defined by $f^*(x) = (r, s, t)$. Let $X = f^*(\mathbb{R}_+^2)$. Suppose for example, $\tilde{A}^{.5} \cap \tilde{B}^{.25} \cap \tilde{C}^0 = \tilde{A}^{.25} \cap \tilde{B}^{.25} \cap \tilde{C}^0$. Then $(.25, .25, 0) \notin f^*(\mathbb{R}_+^2)$. That is, (.25,.25,0) is missing. In this case, $X \subset \langle X \rangle$. Incidentally, f^* is an example of the homomorphism referred to in [9].

We return to the representation of the preferences of three players in Fig. 1 at the beginning of this paper. The maximal set is empty. There are six alternatives in the Pareto set:

$$PS_N(R) = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (.5, .25, 0), (.25, 0, .5), (0, .5, .25)\}.$$

The first three alternatives are elements of $\langle J_{n/2}\rangle$ and are not strictly preferred to any other alternatives. Given the results of this paper, the remaining three alternatives comprise the unique uncovered set:

$$UC(X) = \{(.5, .25, 0), (.25, 0, .5), (0, .5, .25)\}.$$

The model predicts that strategic players in an agenda amendment will pick one of these three alternatives.

Our covering relation C produces an uncovered set $UC(X)$, which with the exception of the trivial case of $\langle J_{n/2}\rangle$, is contained within the Pareto set, $PS_N(R)$. This is a very good result. However, it rests on the assumption that $\langle PS_N(\overline{\rho})\rangle$ denotes the set of alternatives. In other words, there is an alternative for every possible n-tuple of preferences descending from the Pareto set. As it turns out, $UC(X)$ can result in a non-Pareto efficient outcome when immediate predecessors of elements of the Pareto set $PS_N(\overline{\rho})$ are not contained in X, the set of alternatives, a situation we label "vulnerability to holes". This is analogous to the uncovered set under strict preference, which contrary to Miller's conjecture [7] can result in non-Pareto efficient outcomes [8]. We give consideration to this possibility before concluding.

Definition 5.1. Let $x, y \in X$. Suppose xCy. Let $W_{x,y} = \{w \in X | xPw$ and not yPw, $y \neq w \neq (0, 0, 0)\}$. Then (x, y) is said to be vulnerable to holes if not xCy in $X \setminus W_{x,y}$.

Proposition 5.2. *Let $x, y \in X$. Suppose xCy. Then $x \notin \langle J_{n/2}\rangle$.*

Proof. Suppose $x \in \langle J_{n/2}\rangle$. Then there does not exist $w \in X$ such that xPw. Hence not xCy. □

Proposition 5.3. *Let $x, y \in X$. Suppose xCy. Then (x,y) is vulnerable to holes if and only if $y \notin \langle J_{n/2}\rangle$ and not xPy.*

Proof. Suppose (x, y) is vulnerable to holes. Suppose $y \in \langle J_{n/2}\rangle$. Then xCy in $X \setminus W_{x,y}$ since $(0, 0, 0) \notin W_{x,y}$ and not $yP(0,0,0)$ and $xP(0,0,0)$, where the latter condition holds since $x \notin \langle J_{n/2}\rangle$ by the previous proposition. This is contrary to the assumption that (x, y) is vulnerable to holes. Hence $y \notin \langle J_{n/2}\rangle$. It is not the case that xPy else xPy in $X \setminus W_{x,y}$ since $y \notin W_{x,y}$ and not yPy, i.e., y is a w in the definition of C.

Conversely, suppose $y \notin \langle J_{n/2}\rangle$ and not xPy. Suppose xCy in $X \setminus W_{x,y}$. Then there exists $w \in X \setminus W_{x,y}$ such that xPw and not yPw. Hence either $w = (0, 0, 0)$ or $w = y$. Suppose $w = (0, 0, 0)$. Since not yPw, $y \in \langle J_{n/2}\rangle$, a contradiction. Suppose $w = y$. Then xPy, a contradiction. Hence not xCy in $X \setminus W_{x,y}$. □

Theorem 5.4. *Let $x, y \in X$. Suppose xCy. Suppose also that (x,y) is vulnerable to holes. Then $\forall w \in W_{x,y}$, the following conditions hold*:

(1) *If $x \not\succ w$, then there exists a permutation π of $\{1,2,3\}$ such that $x_{\pi(1)} = y_{\pi(1)} > w_{\pi(1)}$, $w_{\pi(2)} > x_{\pi(2)} = y_{\pi(2)}$, and $x_{\pi(3)} > w_{\pi(3)} \geq y_{\pi(3)}$.*

(2) *If $x \succ w$, then* (a) *there exists a permutation π of $\{1,2,3\}$ such that $x_{\pi(1)} = y_{\pi(1)} > w_{\pi(1)}, x_{\pi(2)} = y_{\pi(2)} = w_{\pi(2)}$, and $x_{\pi(3)} > y_{\pi(3)} = w_{\pi(3)}$ or* (b) *there exists a component of w strictly greater than the corresponding component of y and there exists a permutation π of $\{1,2,3\}$ such that $x_{\pi(1)} = y_{\pi(1)} > w_{\pi(1)}, x_{\pi(2)} = y_{\pi(2)} = w_{\pi(2)}$, and $x_{\pi(3)} > w_{\pi(3)} > y_{\pi(3)}$. Conversely, if $w \in X$ and satisfies either* (1) *or* (2), *then $w \in W_{x,y}$.*

Proof. Let $w \in W_{x,y}$. Then $x \neq w$ since xPw. Since $y \notin \langle J_{n/2}\rangle$ by Proposition 5.2, $x \succ y$ by Proposition 4.8. Since xRy and not xPy by Proposition 5.2, two of the components of x equal the two corresponding components of y and the remaining components of x is greater than the corresponding component of y.

(1) Suppose $x \not\succ w$. Since xPw, two of the components of x are strictly greater than the corresponding components of w and the remaining components of w is strictly greater than the corresponding component of x. For simplicity and without loss of generality, we can write either (i) $x_1 = y_1 > w_1, x_2 = y_2 > w_2$ and $w_3 > x_3 > y_3$ or (ii) $x_1 = y_1 > w_1, w_2 > x_2 = y_2$, and $x_3 > y_3, x_3 > w_3$. However (i) does not hold else yPw. Suppose (ii) holds. If $y_3 > w_3$, then yPw, a contradiction. Thus $w_3 \geq y_3$. Hence (1) holds.

(2) Suppose $x \succ w$. Then as in the previous paragraph, we can write without loss of generality, $x_1 = y_1 > w_1, x_2 = y_2 \geq w_2, x_3 > y_3$, and either (i) $y_3 \geq w_3$ or (ii) $w_3 > y_3$. Suppose (i) holds. Then $x_1 = y_1 > w_1$ and since not yPw, $x_2 = y_2 = w_2$ and $x_3 > y_3 = w_3$. In this case, (2) holds. Suppose (ii) holds. Then $x_1 = y_1 > w_1, x_2 = y_2 = w_2$, and $x_3 > w_3 > y_3$ since xPw and not yPw. In this case, (2) holds.

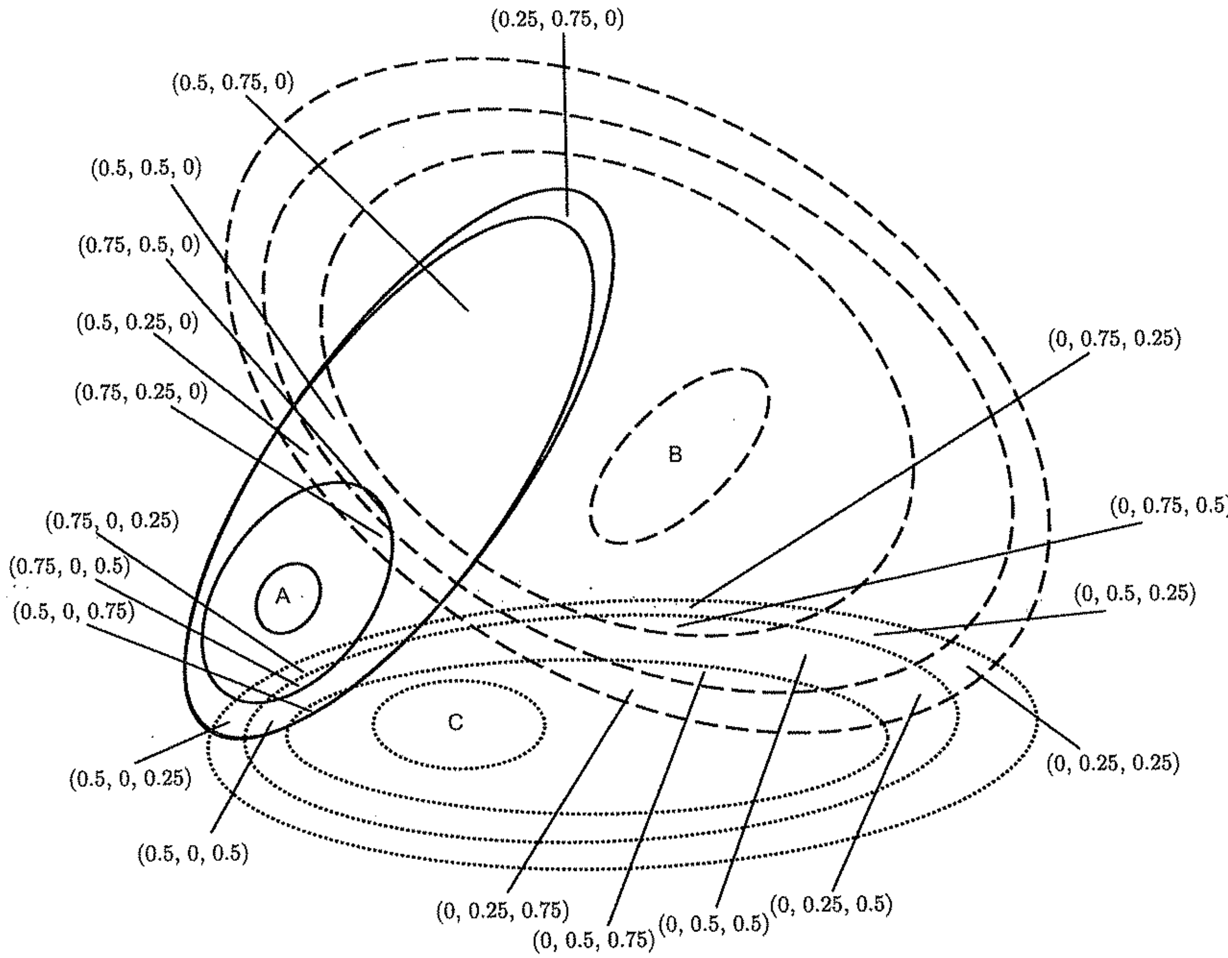


Fig. 2. Player $\tilde{A}$ with congruent preferences at $\alpha = .25$ and .5.

For the converse, the only possible way $w = (0, 0, 0)$ is if (2a) holds, but then $y \in \langle J_{n/2} \rangle$, a contradiction. Clearly $w \neq y$. It is easily verified that xPw and not yPw. □

Both $x \not\succ w$ and $x \succ w$ can only occur when the borders of the α-levels of one or more of the players are congruent at important segments of the policy space. In such cases, the α-levels encompass the same region for a player over at least part of the policy space. This situation is illustrated in Fig. 2. The borders of the α-levels at .25 and .5 for player $\tilde{A}$ are partially congruent.

Fig. 2 depicts holes where $x \not\succ w$ for some $w \in W_{x,y}$. Let $x = (.5, .75, 0)$ and let $y = (.25, .75, 0)$. Then $(.25, 0, .5) \in W_{x,y}$, and $(.5, .75, 0) \not\succ (.25, 0, .5)$. However, (.25,0,.5) is not in X, and x does not cover y.

Fig. 2 also illustrates holes where $x \succ w$ for some $w \in W_{x,y}$. Let $x = (.5, .75, 0)$ and let $y = (.25, .75, 0)$. Then $(.25, .5, 0) \in W_{x,y}$, and $(.5, .75, 0) \succ (.25, .5, 0)$. However, (.25,.5,0) is not in X, and x does not cover y.

We have demonstrated both phenomena using a three-player game. While the simplicity of the three-player game commends it for depicting these situations, it also greatly overstates the likelihood of the occurrence of either. The probability of the occurrence of either $x \not\succ w$ or $x \succ w$ dramatically decreases as the number of players increases. With T^3, a three player game has 125 possible alternatives, a four player game has 625 alternatives, and a five player game has 3125 alternatives. With a high enough N, thousands of alternatives with the right preferences descending from the Pareto set would need to be missing. Thus, in contrast to the uncovered set under strict preference, our uncovered set under fuzzy individual preferences is far less likely to include non-Pareto efficient outcomes. Given that such outcomes are highly unlikely to be chosen by a decisive coalition, this is a good result.

6. Conclusions

Existing definitions of the covering relation return different uncovered sets under thick indifference. Fuzzy preferences are thick. Thus, we developed an appropriate definition of the covering relation for fuzzy preferences. Furthermore, we demonstrated that in the absence of a maximal set, the resulting uncovered set is likely to be contained in the Pareto set. The major exceptions are either unlikely or confined to those elements that are not strictly preferred by majority rule to at least one other alternative. Thus, the uncovered set under fuzzy individual preferences commends itself.

Acknowledgements

This article is a revised version of a paper presented at the 2009 annual conference of the Public Choice Society. We thank the anonymous reviewers and editorial staff of *Fuzzy Sets and Systems* for their invaluable assistance in improving this manuscript. We owe a special debt of gratitude to Lotfi Zadeh, George Klir, and Paul Wang for their encouragement and support of the fuzzy spatial modeling project, of which the research reported in this article is a part. We also wish to express our appreciation to Jay Verkuilen and Joseph Godfrey for their substantive comments on previous versions of this manuscript, Anne Bautch for the many hours she spent designing and solving amendment agendas, and Adam Karnik for helping us to work through the effect of indifference on covering relations. Finally, we gratefully acknowledge the funding provided by an Academic Affairs Research Initiative Grant, the Center for Mathematics of Uncertainty, and the College of Arts and Sciences at Creighton University.

References

[1] G. Tullock, Why so much stability?, Public Choice 37 (1981) 189–202.
[2] J. Sloss, Stable outcomes in majority rule voting games, Public Choice 15 (1973) 19–48.
[3] C.A. Tovey, The instability of instability, NPSOR-91-15, Naval Postgraduate School, Monterey, CA, 1991.
[4] W.V. Gehrlein, F. Valognes, Condorcet efficiency: a preference for indifference, Social Choice and Welfare 18 (2001) 193–205.
[5] L. Ehlers, S. Barberà, Free triples, large indifference classes and the majority rule, Cahiers de recherche 2007-02, Universite de Montreal, Departement de sciences economiques, 2007, available: ⟨http://ideas.repec.org/p/mtl/montde/2007-02.html⟩.
[6] D.H. Koehler, Convergence and restricted preference maximizing under simple majority rule: results from a computer simulation of committee choice in two-dimensional space, American Political Science Review 95 (2001) 155–167.
[7] T. Bräuninger, Stability in spatial voting games with restricted preference maximizing, Journal of Theoretical Politics 19 (2007) 173–191.
[8] T.D. Clark, J.M. Larson, J.N. Mordeson, J.D. Potter, M.J. Wierman, Applying Fuzzy Mathematics to Formal Models in Comparative Politics, Springer-Verlag, Berlin, 2008.
[9] J.N. Mordeson, T.D. Clark, The existence of a majority rule maximal set in arbitrary n-dimensional spatial models, New Mathematics and Natural Computation 6 (2010) 261–274.
[10] N.R. Miller, A new solution for tournaments and majority voting: further graph-theoretical approaches to the theory of voting, American Journal Political Science 24 (1980) 68–96.
[11] K.A. Shepsle, B.R. Weingast, Uncovered sets and sophisticated voting outcomes with implications for agenda institutions, American Journal Political Science 28 (1984) 49–74.
[12] R.D. McKelvey, Covering, dominance, and institution-free properties of social choice, American Journal Political Science 30 (1986) 283–315.
[13] H. Nurmi, J. Kacprzyk, Fuzzy sets in political science: an overview, New Mathematics and Natural Computation 3 (2007) 281–299.
[14] J. Blin, Fuzzy relation in group decision theory, Journal of Cybernetics 4 (1974) 17–22.
[15] J.C. Bezdek, B. Spillman, R. Spillman, A fuzzy relation space for group decision theory, Fuzzy Sets and Systems 1 (1978) 255–268.
[16] J.C. Bezdek, B. Spillman, R. Spillman, Fuzzy relation spaces for group decision theory: an application, Fuzzy Sets and Systems 2 (1979) 5–14.
[17] S.A. Orlovsky, Decision-making with a fuzzy preference relation, Fuzzy Sets and Systems 1 (1978) 155–167.
[18] H. Nurmi, Approaches to collective decision making with fuzzy preference relations, Fuzzy Sets and Systems 6 (1981) 249–259.
[19] J. Kacprzyk, M. Fedrizzi, H. Nurmi, Group decision making and consensus under fuzzy preferences and fuzzy majority, Fuzzy Sets and Systems 49 (1992) 21–31.
[20] J. Kacprzyk, M. Fedrizzi, A 'soft' measure of consensus in the setting of partial (fuzzy) preferences, European Journal of Operational Research 34 (1988) 316–325.
[21] H. Nurmi, J. Kacprzyk, On fuzzy tournaments and their solution concepts in group decision making, European Journal of Operational Research 51 (1991) 223–232.
[22] C. Achen, J. Aldrich, J. Alt, H. Brady, J. Freeman, W. Keech, R.D. McKelvey, R. Morton, C. Simon, H.P. Young, D. Zinnes, The Empirical Implications of Theoretical Models (EITM) Workshop Report. Political Science Program, Directorate For Social, Behavioral and Economic Sciences, National Science Foundation, 2002. Available: ⟨www.nsf.gov/sbe/ses/polisci/reports/pdf/eitmreport.pdf⟩.
[23] H. Nurmi, A fuzzy solution to a majority voting game, Fuzzy Sets and Systems 5 (1980) 187–198.
[24] L.A. Zadeh, Fuzzy sets, Information Control 8 (1965) 338–353.
[25] C.R. Plott, A notion of equilibrium and its possibility under majority rule, American Economic Review 57 (1967) 787–806.
[26] J.M. Enelow, M.J. Hinich, On Plott's pairwise symmetry condition for majority rule equilibrium, Public Choice 40 (1983) 317–321.

[27] J. Greenberg, Consistent majority rules over compact sets of alternatives, Econometrica 47 (1979) 627–636.
[28] R.D. McKelvey, Intransitivities in multidimensional voting models and some implications for agenda control, Journal of Economic Theory 12 (1976) 472–482.
[29] R.D. McKelvey, General conditions for global intransitivities in formal voting models, Econometrica 47 (1979) 1085–1112.
[30] N.R. Miller, In search of the uncovered set, Political Analysis 15 (2007) 21–45.
[31] E.M. Penn, Alternate definitions of the uncovered set and their implications, Social Choice and Welfare 27 (2006) 83–87.
[32] N.R. Miller, Alternative definitions of the covering relation: an extended tour, unpublished manuscript.
[33] G.A. Bordes, M. LeBreton, M. Salles, Gillies and Miller's subrelations of a relation over an infinite set of alternatives: general results and applications to voting games, Mathematics Operations Research 17 (1992) 509–518.
[34] D. Austen-Smith, J.S. Banks, Positive Political Theory II: Strategy and Structure, University of Michigan Press, 2005.
[35] G.A. Bordes, On the possibility of reasonable consistent majoritarian choice: some positive results, Journal of Economic Theory 31 (1983) 122–132.