ELECTORAL SYSTEMS, PARTY SYSTEMS, AND PARLIAMENTARY GOVERNMENT FORMATION

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1. Overview

This module in the *Vote*Democracy course examines interconnections among electoral systems, party systems, and parliamentary government formation. Since these matters exhibit great diversity across nations and considerable diversity over time within many nations, and since the *Vote*Democracy course aims to introduce students to basic elements of the theory of social choice and voting and apply them to contemporary problems of democracy, these interconnected topics are discussed schematically and theoretically, in contrast to the more descriptive treatments typically presented in political science courses on comparative electoral systems.

It is standard to draw a sharp analytic distinction between two types of electoral systems: those based on *Plurality Rule* (or related voting rules) in *single-member districts* (SMDs) and those based on some type of *proportional representation* (PR) in *multi-member districts* (MMDs). The first type, commonly referred to as *First-Past-the-Post* (FPTP), has traditionally been used in English-speaking countries and former British colonies, including the U.K., U.S., Canada, Australia (with some variations), New Zealand (until 1996), India, and many others, and it is often characterized as establishing *majoritarian* democracy. While one can argue with this characterization, it does have two justifications: first, the electoral rule used within each district is majoritarian (as defined in Module 6 and in 3.5 below); second, in parliamentary systems the leading party usually wins a majority of seats and can therefore form a government without coalition partners. The second type is commonly used on the continent of Europe and elsewhere and is characterized as establishing *proportional* democracy. This characterization has two closely related justifications: first, the electoral rule used in each MMD is proportional rather than majoritarian; second, the overall distribution of seats in parliament tends to be approximately proportional to party support in the electorate (which often is not the case under FPTP systems). It is plausible to expect that the nature of an electoral systems shape the nature of party systems — in particular, the number (and therefore the relative size) of political parties that (seriously) compete in elections. Indeed, what is commonly called *Duverger's Law* — perhaps the most famous 'law-like' generalization in conventional political science — asserts that majoritarian electoral systems lead to and sustain two-party systems while proportional electoral systems lead to and sustain multi-party systems.

The next section of this module presents a schematic overview of the origin and nature of political parties and party systems and gives initial consideration to the problem of counting the number of components in a party system. Section 3 notes that electoral systems are based on districts of varying 'magnitude' from which representatives are elected, and it examines the wide variety of electoral rules that may be used within such districts. Section 4 examines how electoral rules create different strategic incentives for voters, candidates, and political parties. Section 5 examines the relationship between votes cast for party candidates and seats won by parties at the national level, with a particular attention to FPTP systems. Section 6 reconsiders Duverger's Law and summarizes recent research that classifies electoral systems in terms of their 'seat-product', on the basis of which characteristics of party systems may be predicted; one implication is that the sharp analytic distinction between FPTP and proportional systems disappears. Section 7 examines the formation of coalition governments in parliamentary systems, as typically occurs in proportional electoral systems.

1.1 Bibliographical Notes and Further Readings

Much of the early work on electoral systems tended to be mainly descriptive; moreover, it was often polemical, arguing the relative merits of majoritarian versus proportional systems. Examples include, Mill (1861), Hare (1859), Finer (1924), Hoag and Hallet (1926), Hermens (1941), Lakeman and Lambert (1955), Van Den Bergh (1956), Mackenzie (1958), and Finer (1975). An up-to-date catalog of the world's electoral systems is provided by Golder (2005) and Bormann and Golder (2013). The first theoretically oriented work drawing on systematic empirical evidence was Rae (1967). Major follow-up works include Katz (1980), Taagepera and Shugart (1989), Lijphart (1994), and Cox (1997). Shugart (2006) provides a comprehensive review of the field of electoral systems research. Edited works with relevant chapters include Bogdanor and Butler (1983), Lijphart and Grofman (1984), Grofman and Lijphart (1986), Colomer (2004), Gallagher and Mitchell (2006), and Herron et al. (2018). Textbook introductions to electoral systems include Amy (2000) and Farrell (2001). Dummet (1997) provides a philosophical overview of the topic of electoral reform. 'Duverger's Law' is due to Duverger (1954). Grofman (2006 and 2016) surveys the effect of electoral systems on political systems beyond the number and size of political parties;

2. Political Parties and Party Systems

This module examines primarily how electoral systems allocate seats to political parties in national parliaments (or other assemblies), rather than how individual candidates are elected. The rationale for this party-centered focus is that political parties are the dominant players in elections, parliaments, and government selection, as well as in the governing process as a whole. Thus it is appropriate briefly to consider why and how political parties acquire this dominance.

2.1 Origin and Nature of Political Parties

Both theory and experience suggest that a system of free elections with large electorates to fill seats in an assembly with governing powers can be expected to lead to the formation of *political parties* and thus to a *party system* of some type. Accordingly, large-scale elections and government formation almost invariably revolve around political parties. This proposition may be dubbed *Schattschneider's Law*, in honor of the classic work on party government by the American political scientist E. E. Schattschneider. In the absence of Schattschneider's Law, Duverger's Law could not arise.

Schattschneider's schematic theory of political party formation is particularly applicable to the formation of political parties in early elections in the United States, since the newly ratified U.S. Constitution created at a specific point in time a new federal government with a popularly elected representative assembly, namely the U.S. House of Representatives (together with an indirectly elected Senate and President), at a time when nationally organized political parties did not exist. In contrast, in Britain and elsewhere parliamentary institutions predate extensive electorates, and both they and political parties evolved gradually over an extended period of time.

In Schnattschneider's view, a *political party* forms as a result of collaboration among *politicians* — that is, elective office holders or office-seekers — rather than among rank-and-file voters. This collaboration extends over geographical areas (e.g., electoral districts and U.S. states), over a wide range of issue areas (in contrast to the more focused collaboration that forms an *interest group* and aims to influence, rather than elect, legislators), and over time (typically enduring over many elections, in contrast to individual candidates), and it aims at securing political power through coordinated electoral activity. The most fundamental kind of collaboration among party politicians is the *candidate selection* (or *nominating*) function, which coordinates the votes of a party's prospective supporters. (See Box XX on Candidate Selection and Primary Elections.)

This theory of party formation starts with a 'pristine' legislature unsullied by any kind of political organization (plausibly approximated by the first U.S. Congress that met in the spring of 1789) that is *elected* by an extensive electorate, that is relatively large, and that makes decisions by voting on a sequence of yes/no votes under *majority rule* (as discussed in Module 6). In such a pristine legislature, political preferences and legislative votes are likely to be highly dispersed and quasi-random. In this circumstance, a small group of members can gain a great advantage by collaborating to form a *legislative caucus* — that is, a group that, prior to any parliamentary vote, meets to agree on a common position and then votes as a bloc one way or the other. So long as other votes remain quasi-random (and thus typically more or less evenly split), such a caucus is likely to carry most votes. But such organization will provoke counter-organization: first the formation of other small caucuses and then the merger of smaller caucuses into larger ones. This process of organization, counter-organization, and merger is likely to continue until one caucus achieves majority status. At this point, the majority caucus has no incentive to expand further because (given legislative majority rule) it is already all-powerful, and any other (minority) caucus is totally shut out of power.

A minority caucus therefore has a strong incentive to upset this state of affairs. Possibly it can induce some members of the majority caucus to defect to its side and thereby achieve majority status within the legislature. But the fact that the legislature is elective — so that there is always an election coming up — gives the minority caucus an incentive to turn itself into an *electoral* (as opposed to merely *legislative*) collaboration — that is, into a *political party*. Its members can pool their resources for electoral battle, both to secure their own reelection prospects and also to nominate and support candidates to oppose members of the majority caucus. The minority caucus can thereby attempt to convert itself into a (majority) political party. Once, the minority caucus undertakes this transformation, the majority caucus must do the same or be electorally defeated, and a party system thus emerges.

One implication of this analysis is that political parties are primarily 'office-seeking' — that is, motivated to win elections, elect their nominees to office, and maximize their share of seats in the assembly. An alternative goal that may be attributed to political parties is that they are primarily 'policy-seeking' — that is, interested in advancing preferred policy goals. But this distinction is not as sharp as it may first appear, as parties need to win elections (or substantial shares of seats in parliament) in order to advance any policy goals. In any case, it does not concern us until the final section on coalition government formation.

2.2 Party Systems

Perhaps the most important single characteristic of a nation's politics and governing process is the degree of fragmentation of its party system — that is, the number of political parties that compete in elections and hold seats in its parliament. The previous discussion of party formation perhaps suggests that a two-party system would emerge (as indeed was true in the United States). But, while it stipulated that members of the legislature were elected, it did not specify the rules by which they were elected, and it is widely accepted that the nature of a party system — and particularly the number of parties — is shaped importantly by the particular rules used to elect members of an assembly.

While it is commonplace in Duverger's Law and elsewhere to make a sharp distinction between *two-party systems*, such as are said typically to exist in majoritarian systems, and *multi-party systems*, such as are said typically to exist in proportional systems, this simple numerical distinction is rather unsatisfactory, as is suggested by the data in Tables 1-10 that show, for a variety of elections in a variety of countries, the number of votes received by candidates of each party, together with the number of seats in parliament (or equivalent assembly) each party won.

The United States has the world's most definitively two-party system, probably due not only to its FPTP electoral system but also to other constitutional features — in particular, its presidential elections and unique system of primary elections (see Box XX) that allows, or even encourages, dissident factions to compete within one or other major party rather creating additional parties. Table 1A shows results for recent elections for the House or Representatives — the elections most comparable to those presented in other tables. Table 1B shows results for recent presidential election (plus 1984, the most recent 'landslide' presidential election), where 'presidential electors' correspond to 'seats' in other tables (see Box XX on the U.S. Electoral College); it also shows the allocation of electors that would result if they were elected as individuals from the same SMDs that elect House members.

Table 2A shows results British general elections from 1924 through 1931 during which the British party made a transition from a Conservative-Liberal two-party system to a Conservative-Labour one. Table 2B shows results British general elections from 1950 through 1966 when, except for some persistent support for the Liberal Party, the U.K. had a two-party system almost as dominant as that in the U.S. Table 2C shows the results of the 2010 election, by which time the rise of additional parties produced a 'hung parliament' in which no party controlled a majority of seats. Table 3 shows the results of Canadian elections from 1963 through 1984, showing that a strictly two-party system never existed despite the FPTP electoral system. Table 4 shows the results of elections in New Zealand from 1978 through 1996, the first election after the country switched from an FPTP system to a variant of proportional representation.

Table 5 shows the results of a recent election in Ireland, which uses the quasi-proportional 'Single Transferable Vote' rule in small (3-5 member) districts to fill parliamentary seats, while Tables 6-10 show elections results from selected elections in five countries — Italy (though it has since changed its electoral system), Germany, Denmark, Netherlands, and Israel — that use some variant of 'party list proportional representation'. (These electoral rules are discussed in the next section.)

While it seems uncontroversial to say that the United States, the United Kingdom from 1950 to through 1966, and New Zealand before 1996 have two-party systems, the cases of Canada and the

United Kingdom more recently are more questionable. At the same time, while all the countries with proportional systems can fairly be said to have multi-party systems, they exhibit considerable variety with respect the number (and relative sizes) of these multiple parties, and the question arises whether characteristics of their electoral systems more specific than simple ‘proportionality’ help account for this variation.

Of course, party systems have characteristics beyond the number and relative sizes of the parties. These include the relationship between parties and groups in society created by ‘social cleavages’ such as social class, religion, language, race or ethnicity, urban versus rural residence, and so forth, as well as the relative positioning of the parties with respect to matters of policy or ideology and their national versus localistic orientation. It is worth observing that, in so far as all nations are more or less divided in these respects, both parties in a two-party system are necessarily heterogeneous and internally somewhat factionalized, while parties in multi-party systems (especially highly fragmented ones) may be relatively homogeneous, often representing rather specific groups or policy goals. However, such considerations are largely beyond the scope of this module.

2.3 Bibliographical Notes and Further Readings

The theory of party formation presented here is due to Schattschneider (1942), Chapter 3; for related presentations, see Aldrich (1995), especially Chapter 2, and Colomer (2007). On party systems more generally, see especially Sartori (1976) and Mair (2002). Many more national election results for many countries may be found on Wikipedia or the websites of their national assemblies or election authorities.

3. Electoral Rules

An *electoral system* determines how members are elected to a representative assembly, be it a small local council or a large national parliament, and it has two principal components: a system of (almost always) geographically defined *electoral districts* and an *electoral rule* employed within each district. The entire system is shaped by overall *assembly size* — that is, the total number of seats in the assembly.

3.1 Assembly Size and District Magnitude

A basic feature of an electoral system is *assembly size* (i.e., the number of seats), which we designate by S. Other things equal, larger assemblies allow for representation of smaller parties and more generally can better realize representation of parties (and other groups) proportionate to their size in the population. Typically larger assemblies are used to represent larger populations; thus provincial and state assemblies are usually larger than city or county councils but smaller than national parliaments, and more populous nations tend to have larger parliaments than less populous ones.[1]

[1] In 6.3 we take note of a specific quantitative relationship between expected assembly size and population size.

Districting assures a measure of geographic representation. For example, given a five-member assembly (such as a local council), each member may be elected from a *single-member district* (SMD), so that the territory represented by the assembly is divided into five districts (typically with approximately equal populations) and one council member is elected from each district. But if geographic representation is deemed unimportant or even undesirable (as it may encourage members to focus on the interests of their districts rather than the whole community), members may be elected *at-large*, so that all five members are elected from a single area-wide *multi-member district* (MMD). Another possibility is a *mixed system* in which some members are elected from SMDs and others are elected at-large. In a larger assembly representing a larger population spread over an extensive area, members may be elected from several or many MMDs. This may be implemented in either of two ways. First, the region or nation may be divided into approximately equally populated districts, each of which elects the same fixed number of members. Alternatively, it may be divided into unequally populated districts (perhaps defined by the boundaries of states, provinces, counties, or administrative regions), each of which elects a number of members that is approximately proportional to its population.[2] The number of assembly members elected from a district is called the *district magnitide* and is designated by M. Since magnitude may vary from district to district in a given electoral system, M may also refer to *average* (mean) *district magnitude* in the system as a whole. In any event, the ratio S/M is equal to the number of districts k. Given SMDs, $M = 1$ and $k = S$; given election at-large $M = S$ and $k = 1$. The special case of $M = S = 1$ covers the election of a single president, governor, mayor, or some other single official.

3.2 Electoral Rules and Ballot Types

Next, some *electoral rule* must be used to elect one or more members from each district. Section 6 of the previous module describes various voting rules used to elect a single candidate, which may therefore be used as electoral rules for SMDs. The most commonly used is *Plurality Rule*. As already noted, the electoral system that combines SMDs with Plurality Rule is called *First-Past-the-Post* and is used in most English-speaking countries. But other single-winner rules discussed in Module 6 may be (and sometimes are) used in conjunction with SMDs; these include *Plurality Runoff Rule* (used in some U.S. elections and French presidential elections); the *Alternative Vote* (used to elect members of the Australian House of Representatives but rejected by British voters in a 2011 referendum), also known in the U.S. as *Instant Runoff Voting* or *Ranked Choice Voting*; and *Two-Round Majority-Plurality Rule* (used in French parliamentary elections). In principle, *Approval Voting* and *Borda Rule* could be used as electoral rules in SMDs but rarely if ever are.

These and many other electoral rules are *candidate-oriented*, in that a voter is presented with a ballot that lists a number of candidates (often showing their party affiliations) and asks the voter to express some kind of preference with respect to these candidates. In practice, SMD systems always use candidate-oriented ballots. Plurality Rule, Plurality Runoff rule, and Two-Round Majority-Plurality (as well as Approval Voting) use *nominal ballots* on which a voter either votes 'for' a candidate (typically by putting an 'X' beside the name) or not. In contrast, the Alternative

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[2] Districting is sometimes based on electorate size rather than population size. Module 8 provides a thorough discussion of electoral districts and issues related thereto.

Vote (and Borda Rule) require an *ordinal ballot* on which a voter ranks the candidates in order of preference (typically by putting a '1', '2', etc., beside the names of candidates to indicate order of preference).

Some MMD systems also use candidate-oriented ballots but most use *party-oriented* ballots, in that a voter is presented with a ballot that lists a number of parties and asks the voter to express a preference with respect to these parties. In practice, such ballots are nominal, in that each voter simply votes for one party in the manner of Plurality Rule for candidates. Proportional representation electoral rules are party-oriented in this sense, as their purpose is to give each party a share of seats in parliament that matches its share of votes. However, some proportional representation ballots allow voters to express preferences for candidates as well.

3.3 Candidate-Oriented Electoral Rules for MMDs

We first discuss electoral rules that use candidate-oriented nominal ballots; they are typically applied in small-magnitude MMDs (e.g., $2 \leq M \leq 5$). We use the following notations:

M = the number of candidates to be elected (i.e., the district magnitude);

K = the maximum number of votes each voter can cast (typically the maximum number of candidates each voter can vote for);

V = the total number of voters casting ballots.

Under ordinary Plurality Rule, $K = M = 1$. Plurality Rule can be generalized to MMDs in either of two ways: increasing K to match the larger M or holding K at 1 (or in any case using some K such that $1 < K \leq M$). These two possibilities produce electoral rules with fundamentally different properties.

The first option (i.e., $K = M$) produces variants of what may be called *MMD Plurality Rule* systems. The most common is *Plurality At-Large Rule*, while a less common variant is *Plurality Block Rule*; in either case, a voter *may* vote for as many as M candidates but in the latter case a voter *must* do so (or else the voter's ballot is void). Plurality At-Large Rule is used in some local council elections in both the U.S. and U.K., as well as in some state legislative elections the U.S. An MMD variant of AV (called *Preferential Block Voting*) was used to elect members of the Australian Senate prior to 1948. A theoretically interesting variant of MMD Plurality is what we may call (though the term is not standard) *Party-List Plurality Rule*, under which each party runs a list (or 'slate' or 'general ticket') of M candidates and, using a party-oriented ballot, voters choose among party lists (rather than individual candidates), and one list is chosen *as a whole* on the basis of Plurality Rule; this turns an MMD election into one that is logically equivalent FPTP where the number of seats is scaled up by a factor of (the average value of) M.[3] Approval Voting and Borda Rule can likewise be generalized to MMDs by electing the top M candidates ranked by approval votes or Borda scores.[4]

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[3] The best (and perhaps only) example of Party-List Plurality Rule is provided by the 'winner-take-all system' used by almost all U.S. states to elect presidential electors. (See Box XX on the U.S. Electoral College.)

[4] Several ministates in the South Pacific use variants of MMD Borda Rule.

The second option with $K < M$, under which a voter may vote for several candidates but fewer than the number to be elected, is known generally as *Limited Voting*. The important special case of $K = 1 < M$ produces the *Single Non-Transferable Vote* (SNTV), which has been used in parliamentary elections in East Asian countries.

One other multi-winner electoral rule that uses nominal ballots may be mentioned. Under *Cumulative Voting* (CV), each voter casts as many votes as there are candidates to be elected (so $K = M$) but, instead of being spread over M different candidates, the M votes may be 'cumulated' on fewer than M candidates and perhaps all 'plumped' a single candidate. A few local councils in the U.S. and some corporate and other organization boards are elected by CV.

The principal candidate-oriented electoral rule (other than Borda Rule) that employs an ordinal ballot in small MMDs is the *Single Transferable Vote* (STV), a rather complicated rule used in Ireland, Malta, Australian Senate elections since 1948, and in a few other nations and localities. In the U.K., STV has been advocated by the Electoral Reform Society (ERS) and by the Liberal (now Liberal Democrat) Party for many years. It works in the following manner.[5]

(1) The total number of ballots cast (V) is determined.

(2) The *Droop quota* Q_D is calculated, where Q_D is the ratio $V/(M+1)$ rounded up to the next whole number. It can be checked that Q_D is the smallest number of (first-preference) votes such that no more than M candidates can get that number votes. A candidate with at least Q_D votes is said to 'meet quota'.

(3) The ballots are sorted into piles according their first preferences. Any candidate who meets quota is elected. If M candidates meet quota, counting stops.

(4) Otherwise, 'surplus' ballots are transferred from elected to non-elected candidates. Ballots in excess of Q_D in the piles of elected candidates are transferred to the piles non-elected candidates according to the second (or lower) preferences expressed on these ballots.[6] As a result of the transfer of surplus votes, additional candidates may now meet quota and be elected. If M candidates now meet quota, counting stops.

(5) Otherwise, the candidate with the *fewest* (first-preference plus transferred) ballots is eliminated and all of his or her ballots are transferred according to the highest preference expressed on the ballots for remaining (non-elected and non-eliminated) candidates. As a

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[5] The following describes the ERS rules for STV, which are designed for hand counting of ballots. Though complicated, this description does not address all the complexities (resulting from incomplete or 'truncated' rankings on ballots, various kinds of ties, etc.) that can arise in the STV ballot counting process. Still more complicated STV variants that require computer processing of ballots have been proposed (and can produce different sets of winners).

[6] However, since it is arbitrary *which* specific ballots are deemed to be surplus and different ballots have different second (and lower) preferences, ERS recommends that votes be transferred in proportion to the second (or lower) preferences expressed on *all* the ballots that elected a candidate; this means that votes transferred from elected to unelected candidate typically have fractional values.

result of this transfer, additional candidates may meet quota and be elected, with their surplus votes transferred as above. If M candidates now meet the quota, counting stops.

(6) Otherwise, the process of sequentially eliminating candidates with the fewest votes and transferring their votes continues until M candidates have met quota and are elected.

Note that if $M = 1$, $Q_D = V/2$ rounded up to the next whole number, i.e., a simple majority of votes cast, and ballots are transferred only from eliminated candidates, so the Alternative Vote is the special case of STV applied to SMDs.

3.4 Party-Oriented Electoral Rules for MMDs

Essentially the only type of party-oriented electoral rule (other than Party-List Plurality Rule) is *Party-List Proportional Representation*. Varieties of list-PR are used in most continental European countries, Israel, and many other countries. Its rules can be extremely complex with many possible variations. No two national PR systems are identical and individual PR countries frequently modify the details of their electoral rules.

We use the following additional notation in discussing apportionment formulas:

V_i = the number of votes for party i (where $\Sigma V_i = V$);
v_i = the vote share of party i expressed as either a fraction or percent (so $\Sigma v_i = 1$ or 100%);
S_i = the number of seats awarded to party i (where $\Sigma S_i = S$);
S^*_i = party i's *ideal quota* of seats, where $S^*_i = v_i \times M$; and
s_i = the seat share of party i expressed as either a fraction or percent (so $\Sigma s_i = 1$ or 100%).

Note that a party i's ideal quota S^*_i is the 'quantity' (as opposed to number) of seats it would receive if seats could be divided so that seat shares precisely match vote shares. However, such quotas almost always have fractional values, whereas seats must be awarded in whole numbers.[7]

Under Party-List PR, candidates are elected from (typically fairly large) MMDs — and possibly a single nationwide MMD (so $M = S$) — by voters who indicate their preferred party by means of a nominal ballot. Each party has an *ordered list* of M candidates. Once each party's vote has been counted, a mathematical *apportionment formula* is used to allocate the M seats among the parties in as close proportion to their vote shares as possible.[8] If the apportionment formula awards a party s_i seats, the top s_i candidates on i's list are elected. But the catch is that, since parties cannot be awarded fractional seats, perfect proportionality can almost never be achieved. If the number of seats to be allocated is small (as in small MMDs or small local councils), proportionality is unavoidably likely to be quite crude.

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7 It might be asked why each party cannot simply be awarded its ideal quota rounded to the nearest whole number. But this does not always work because the seats so awarded may not add up to M (in the same way percentages in tables often carry a note saying 'Does not add up to 100% due to rounding').

8 Such formulas were first devised to carry out the requirement of the U.S. Constitution that 'Representatives . . . shall be apportioned among the states according to their respective numbers'.

More fundamentally, the phrase 'in as close proportion to vote shares as possible' is ambiguous. Table 11 show a particular profile of vote shares for three parties together with their ideal quotas,[9] and identifies the *only* ways of apportioning five seats among three parties such that, if party A has a larger vote share than party B, A receives no fewer seats than B.[10] While apportionments (1) and (5) satisfy this condition, we would pretty clearly reject them as deviating greatly from proportionality given the vote shares specified. At the same time, we would have a hard time deciding which of the apportionments (2), (3), and (4) is 'in as close proportion to vote shares as possible'. In fact, we will see that three commonly used apportionment formulas applied to this example produce each of these three different apportionments.

Apportionment formulas are of two main types.[11]

Quota (or *Largest Remainder*) *methods*. These methods award each party *i* one seat for every 'quota' that its vote V_i contains and then assign any remaining seats to the parties with the 'largest remainders', where a party's *remainder* is the difference between its vote and the whole number of quotas its vote contains. Different methods use different quotas: the *simple* or *Hare* quota $Q_H = V/M$, the *Droop* quota $Q_D = V/(M+1)$ discussed in connection with STV, and the *Imperiali* quota $Q_I = V/(M+2)$, where each ratio is rounded up to the next whole number. Taking them in the order listed, the quotas become smaller and therefore allocate more seats, leaving fewer to be allocated on the basis of remainders; the effect is to make the allocation of seats more favorable to larger parties. The most common quota apportionment method, *Largest Remainder-Hare* (LR-H), uses the Hare quota and may most straightforwardly be described in this way: first give each party *i* its ideal quota S^*_i rounded down to the nearest whole number and then award any remaining seats to the parties with the largest remainders in their ideal quotas.

Applying LR-H to the example in Table 11, we first give each party its ideal quota rounded down (3 for P_1, 1 for P_2, and 0 for P_3), which leaves one seat unallocated; since its ideal quota has the largest remainder (0.450), P_3 is awarded the fifth seat, thereby producing apportionment (4). Tables 12 illustrates all three quota methods applied to a five-party vote profile with $M = 5$.

Further examination of the example in Table 11 reveals a paradoxical feature of LR-H (and other quota methods). Suppose the number of seats available is increased from 5 to 6: the ideal quotas are now 3.78, 1.68, and 0.54, so as before the parties are initially awarded 3, 1, and 0 seats; but P_1 and P_2 now have the largest remainders and are awarded the two remaining seats. Thus the effect of increasing the number of seats available is to deprive P_3 of the one seat it won with $M = 5$.[12]

9 Note that ordinary rounding of the ideal quotas leaves one seat unallocated

10 That is, these are the only apportionments of five seats among three parties that satisfy 'monotonicity condition' M1 given in 3.6.

11 Apportionment formulas, and proportional representation generally, are examined in considerably greater detail in Module 11.

12 This surprising feature of LH-R is commonly called the 'Alabama paradox'; the name results from its discovery in the context of U.S. congressional apportionment following the 1880 census.

Divisor (or *Highest Average*) *Methods*. While quotas must be calculated for each district magnitude, the standard description of divisor methods allocates seats in a priority order from 1 to M — that is, divisor methods determine which party has the strongest claim to the first seat, which then has the strongest claim to the second seat, and so forth. For each seat it has been awarded, a party's vote is adjusted downward — and the strength of its claim on the next seat is thereby reduced — by dividing it by a sequence of divisors. The *D'Hondt* method uses the divisor sequence 1, 2, 3, 4, etc., the *Sainte-Laguë* method uses the sequence 1, 3, 5, 7, etc., and the *Modified Sainte-Laguë* uses the sequence 1.4, 3, 5, 7, etc. Before any seats are awarded, the votes for *all* parties are divided by the first number in the sequence. Whatever the first number, this operation clearly does not affect the relative claims of the parties to the first seat, which under any divisor (or quota) method is awarded to the party with the largest vote. This party's vote is then divided by the second number in the sequence to produce its adjusted vote, and the second seat is awarded to the party with the largest (adjusted) vote. Thus, D'Hondt awards the leading party the second seat as well as the first in the event that its vote exceeds that of the runner-up party by a ratio of 2 to 1; otherwise the second seat goes to the runner-up party. Under Sainte-Laguë, the critical ratio is 3 to 1, while under the modified version it is 3 to 1.4 or about 2.14 to 1. At each point in the sequence of awarding seats, the next seat is awarded to the party with the largest adjusted vote. D'Hondt treats large parties more favorably than Sainte-Laguë does, while Modified Sainte-Laguë stands between them in this respect.[13]

Applying D'Hondt to the example in Table 11 gives P_1 the first and second seats, P_2 the third, and P_1 the fourth and fifth, thereby producing apportionment (2). Applying Sainte-Laguë, gives P_1 the first seat, P_2 the second, P_1 the third and fourth, and P_2 the fifth, thereby producing apportionment (3). Applying Modified Sainte-Laguë gives P_1 a second seat before giving P_2 its first but also ends up producing apportionment (3). Thus, the LR-H, D'Hondt, and Sainte-Laguë (modified or not) formulas produce each of the three plausible apportionments. (The bottom half of Table 13 will be discussed later.)

Table 13 illustrates the D'Hondt, Sainte-Laguë, and Modified Sainte-Laguë methods applied to a five-party profile of vote shares and shows seat allocations for $M = 2$ through $M = 9$. Note that the latter two apportionments converge once $M = 9$, and clearly they are identical once every party has been awarded one seat.

The fact that divisor methods award seats sequentially as M increases means that they cannot exhibit the paradoxical feature of LR-H. However, D'Hondt exhibits a different possibly paradoxical feature. Note that P_1's vote share is 0.4378, so its ideal quota with $M = 6$ is 2.627. We might therefore expect that P_1 would be awarded either 2 or 3 seats — that is, its ideal quota rounded either

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[13] More precisely, it harder for small parties to win their first seats under Modified Sainte-Laguë than under the unmodified version but thereafter the two formulas operate in the same way. Divisor methods for a fixed and large M may be more readily implemented by dividing each party's vote by a common divisor such that, when the resulting quotients are rounded to whole numbers according to some rule, they add up to M. Divisor methods differ according to the rounding rule used: D'Hondt rounds all quotients down to the next whole number, while Sainte-Laguë rounds quotients up or down in the normal manner. Any number within a certain range can qualify as the common divisor; while it must be discovered by trial and error, the divisor will be close to V/M.

up or down — but in fact D'Hondt awards it 4 seats. (Other divisor methods can also exhibit this anomaly, but are less likely to do so.) In contrast, LR-H by design always 'stays in quota'.

Though based on quite different principles, LR-H and (unmodified) Sainte-Laguë allocate seats in very similar (often but, as Table 13 shows, not always identical) ways, and appear to exhibit no bias between larger and smaller parties. On the other hand, D'Hondt tends to favor larger parties (as Tables 13 and 14 suggest).

When an apportionment formula is applied to a MMD, a small party may fail to win even one seat. However, given more than two parties, it is not possible to specify a specific vote share or *threshold* that is both necessary and sufficient for a party to win a seat. This is because the number of seats a party wins depends, not only on its own vote share, but also on how the remaining votes are shared among other parties. However, it is possible to specify bounds on such a threshold, which vary by formula. The lower bound, or *threshold of representation*, is the vote share above which a party *may* win a seat but below which it certainly does not. The upper bound, or *threshold of exclusion*, is the vote share above which a party *certainly* wins a seat but below which it may not. Such thresholds for different apportionment formulas can be specified by algebraic expressions that always depend (inversely) on district magnitude M and often on the number of parties N as well. Moreover, expressions for the threshold of exclusion may differ according to the magnitude of N in relation to M. While we will not present these sometimes complicated expressions here, several points may be noted.

First, in the event that $N = 2$, the thresholds of representation and exclusion are identical. Second, in the event that $N > M$ (more parties than available seats), the threshold of exclusion is $1/(M+1)$ for all apportionment formulas. Third, in the event that $M = 1$, all apportionment formulas entail the same threshold of representation (namely, $1/N$) and the same threshold of exclusion (namely 1/2), which are just those entailed by Plurality Rule; this reinforces the point that PR rules are equivalent to Plurality Rule in the SMD case. Finally, consideration of the simple case of Plurality Rule helps clarify the nature of the two thresholds. If a party wins more than $1/N$ of the vote, it will win the seat if all other parties happen to win equal vote shares, but it certainly cannot win the seat if it wins less than $1/N$ of the vote (i.e., less than the average party vote share). If a party wins more than 1/2 of the vote, it certainly wins the seat but it may also win the seat with less than 1/2 of the vote (if two or more other parties split the remaining vote).

Variations in list PR electoral systems include the following details noted in order of their practical importance.

Legal thresholds. Some district-level and most national PR systems impose a fixed *legal threshold* (distinct from the thresholds discussed just above) such as 1.5%, 3%, or 5% of the total vote that a party must meet before it qualifies for any seats, even though the normal operation of the apportionment formula might award it one or more seats. The apportionment of seats among parties that meet the legal threshold is then based, not on their shares of the total vote, but on their shares of the total vote cast for parties that meet the threshold.

MMD magnitudes and tiers. Different PR systems use different and often varied magnitudes of MMDs. Moreover, many use two or more tiers of MMDs, for example small local MMD together with larger regional MMDs. Some systems provide for a tier of *national adjustment seats* to be allocated among parties so as to bring the overall allocation of seats in parliament as close as possible to what would be produced with a single nationwide MMD, even while most members are elected from multiple smaller MMDs. In addition, different apportionment formulas may be used in different tiers and, if quota rules are used, remainders from lower tiers may be applied in higher tiers.

Candidate Preferences. While party lists of candidates are often *closed*, some types of list-PR provide for (in varying degrees) *open* lists such that voters can express preferences for certain candidates, thereby (possibly) influencing which particular candidates are elected. Some systems allow voters to vote directly for one or more individual candidates, with these votes counting for the candidates' party as whole as well as for the individual candidates.[14]

'Reinforced' PR. An otherwise PR system may award a party winning a majority of votes in a given district all *M* seats, or award the party winning the plurality of votes nationwide a fixed seat bonus, in order to make it more likely that the leading party wins a majority of seats and can form a one-party government. Such variants, which intentionally depart from proportionality, have been used in France, Italy, Greece, and elsewhere.

Party Alliances. Some PR systems allow two or more parties to form a pre-election alliance (or *apparentement*) by pooling their lists and votes in order to (possibly) increase the number of seats they jointly win.

These variations, though sometimes important in practice, will for the most part not be further considered in the schematic and theoretical overview of electoral systems presented here. In particular, our focus is on how electoral systems allocate seats to parties, not on which individual candidates fill these seats.

3.5 Mixed Electoral Systems

Some countries use 'mixed' electoral systems that combine FPTP and PR in various ways, for example by electing about half the members from SMDs by Plurality Rule and the other half by list PR in large MMDs. The PR component either may be *parallel* to the FPTP component or may *compensate* for disproportionality in the FPTP component. In the first case, the PR apportionment formula is applied only to the seats elected by PR; the effect is that the overall distribution of seats is a compromise between the FPTP and PR results. In the second case, the PR apportionment formula is applied to the whole number of parliamentary seats and the PR seats are allocated so as to bring the overall distribution of seats in line with the PR results (subject to any legal threshold) in the manner of a very large tier of national adjustment seats; the effect is that, even as about half the members are tied to local SMDs, the system is equivalent to PR with respect to the overall allocation of seats to parties.[15] A further question is whether each voter casts only a single vote for

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[14] Recall 7.4 of Module 6.

[15] However, the FPTP results may be so disproportional that compensation is mathematically impossible within the normal assembly size. In this case, assembly size is sometimes temporarily increased with 'overhang seats'.

the local SMD candidate listed on the ballot by party affiliation, which is also counted as the party-oriented vote in the PR election, or whether voters can 'split' their votes by supporting a candidate of one party in the SMD election and a different party in the PR election. The postwar (West) German constitution notably provided for a mixed electoral system with compensating PR, and since 1953 German voters have been presented with both a candidate-oriented ballot for local SMD candidates and a party-oriented ballot for statewide party lists and thus are able to 'split' their votes, producing what is sometimes called *Personalized PR*. More recently New Zealand has adopted a similar system (also called the *Additional Member System*) for its national elections, as has Scotland for election of its devolved parliament.

3.6 Properties of Electoral Rules: Majoritarian and Proportional

All SMD electoral rules in actual use are *strictly majoritarian* in the sense defined in Module 6 — that is, they give any majority coalition of voters the power to elect its preferred candidate. Moreover, they are *simply majoritarian* — that is, the members of a majority coalition can elect its preferred candidate simply by casting all their votes for (or placing at the top of all their ordinal ballots) this candidate. Thus the candidate of a party preferred by a majority of voters in the district is likely to be elected, though if several candidates associated with the majority party are running for the single seat, coordination failure may allow a candidate of a minority party to win the most votes and be elected under FPTP (as suggested by the discussion of 'clone candidates' in Module 6, section 7.4).

In like manner, all variants of MMD Plurality Rule (as well as MMD Approval Voting) are *majoritarian*, in that they give any majority coalition of voters the power to elect *all M* candidates. This implies that a party supported by a majority of voters in a district can nominate *M* candidates and expect (with varying degrees of confidence, depending on the particular rule) to elect all of them. Conversely, even a large minority party cannot be assured of electing any candidates, though it may succeed in doing so in the event of vote splitting or coordination failure within the majority party. Compared with At-Large Plurality Rule, Plurality Block Rule slightly mitigates coordination problems, while Party-List Plurality Rule precludes them entirely.

Note, however, that this discussion pertains to the majoritarianism of *electoral rules* operating *within* districts, not to FPTP or MMD Plurality Rule *electoral systems* operating nationwide. Under such systems, a nationwide majority may be distributed across districts in such a way that it cannot elect a governing majority of members to the national assembly. The simplest possible example is provided by a FPTP system with just three districts with equal numbers of voters. A national majority of 60% composed of 45% of the voters of districts in 1 and 2 and 90% of the voters in district 3 does not have the power to elect a majority of the three assembly members, and is therefore a victim of what is called an *election inversion*, an *election reversal*, or a *referendum paradox*.[16] In contrast (though with minor qualifications noted below), list-PR electoral systems empower any national majority of voters to elect a governing majority in the assembly.

[16] Stated more directly, a two-party FPTP election produces an inversion when one party win a majority of votes nationwide but the other party wins a majority of seats in the assembly. The U.S. 2000 and 2016 presidential elections provide examples (where electoral votes correspond to seats). This phenomenon was noted in Module 6 and is discussed further in section 6.2 of this module.

In contrast to MMD Plurality Rule variants, the other electoral rules for MMDs discussed here are by design *non-majoritarian* and (with the partial exception of Limited Voting) are instead *proportional* — that is, they do not allow a majority of voters to elect all M members in the district but rather allow any bloc of voters to elect a number of candidates of their choice approximately proportional to the bloc's share of the electorate. However, the degree of proportionality that it is logically possible is clearly limited in small MMDs and strategic and coordination problems arise under some proportional rules.

Under STV, a party's success in electing candidates within a MMD depends on the number of Droop quotas contained in its electoral support contains, and this is more or less true under SNTV and CV as well. For example, in an MMD with $M = 5$, $Q_D \approx V/6 \approx 17\%$ of the vote, so a party with at least this much support should be able to win one seat in the MMD, one with about 34% should be able to win two seats, and so forth. Let's see how this works under specific electoral systems.

Under SNTV, a party with support clearly less than two full Droop quotas should nominate a single candidate for whom its supporters would cast their single votes. If the party has a full quota of supporters, this candidate will be among the top M candidates, since no more than M candidates can receive such a quota of votes. However, a party that has greater electoral support faces strategic problems. The first is how many candidates it should nominate: too few and the party will certainly not receive its roughly proportionate share of seats; too many, and it risks winning fewer than its share, and perhaps none at all, if its supporters spread their votes over too many candidates, many or all of whom therefore fail to be among the top M. The second problem is that, even if it nominates the appropriate number of candidates, the votes of the supporters of a party with two or more full quotas of support must be coordinated so that they are spread as equally as possible over all of its candidates (thereby maximizing support for its least supported candidate). As a result, SNTV tends to produce somewhat 'subproportional' results, such that small parties win more than their proportionate share of seats and big parties less, because big parties face more severe strategic and coordination problems. Cumulative Voting has essentially the same properties as SNTV, except that the voter coordination problem is mitigated by the fact that party supporters can individually distribute their M votes more or less equally among the party's candidates.[17]

However, STV essentially eliminates these strategic problems in that, if every one of h voters ranks (in any order) a set of candidates higher than all candidates outside of the set, at least one of these candidates will be elected for every full quota contained in the set of h voters — that is, at least h/Q_D (rounded down to the nearest whole number) candidates. Indeed, STV can in principle be characterized as a means of implementing the principle of 'free association' (or 'self-defined constituencies'), according to which any group of voters of size Q_D is empowered to elect a candidate of its choice, regardless of its geographical distribution over the MMD (which could in principle be the whole nation). But as a practical matter STV can be implemented only in small magnitude MMDs, so realization of this principle is quite constrained in practice.

[17] Some variants of CV allow voters to cast fractional votes, so that each voter can distribute them precisely equally among a set of candidates.

We characterize SNTV, CV, and STV as *quasi-proportional* electoral rules because they produce more or less proportional outcomes but at the same time these rules (i) are not based on any explicit apportionment formula, (ii) do not achieve proportional results entirely reliably (as they depend on the behavior of voters and parties), and (iii) in practice can be used only in small MMDs that do not permit a high degree of proportionality. Limited Voting lies between variants of MMD Plurality Rule ($K = M$) and SNTV ($K = 1$) and produces a (weighted) compromise between majoritarian and proportional results (where the weighting depends on the magnitude of K relative M) such that the leading party likely wins fewer than all the seats but more than its proportionate share.

Partly because they use explicit apportionment formulas but mostly because they can be used in large MMDs and do not entail strategic or coordination problems for parties that can expect to meet any legal threshold, party-list PR systems can produce highly proportional results. But just as single-winner voting rules run into problems once there are three or more candidates (as discussed in Module 6), PR apportionment formulas run into various problems once there are three of more parties. However, many of these problems may be deemed relatively minor.

First, we identify a number of *monotonicity* conditions that say, in one way or another, that 'more votes should give more (or at least no fewer) seats'. While we might expect all apportionment formulas to satisfy all of these conditions, in fact all fail some of them.

(M1) *Votes-Seats Monotonicity with Respect to Parties*: if party A wins more votes than party B, A should win no fewer seats than B.

In the absence of *apparentement*, every apportionment formula satisfies this condition. Recall that we applied M1 to identify the apportionments in Table 13. However, as discussed in Section 7, PR makes party coalitions important in government formation, so the following more general condition becomes relevant.

(M2) *Votes-Seats Monotonicity with Respect to Coalitions of Parties*: if one coalition of parties collectively wins more votes than another coalition, the former should collectively win no fewer seats than the latter.

Given three or more parties, no apportionment formula always meets this condition. This means that (coalitionwise) election inversions are possible even under list-PR — that is, a coalition of parties can control a majority of seats even though the complementary set of parties won more votes.[18]

(M3) *Votes-Seats Monotonicity with Respect to a Party in Successive Elections*: if party A's vote share increases from one election to the next (while assembly size remains constant), the number of seats A wins should not decrease.

Given three or more parties, no apportionment formula always meets this condition, because the number of seats a party wins depends not only on its own vote share but on how the remaining votes

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[18] While this can most obviously be true if the complementary set includes parties that won votes but no seats, it can also be true even if vote shares are based on seat-winning parties only.

are shared among the other parties (hence the distinction between thresholds of representation and exclusion).

(M4) *Votes-Seats Monotoncity with Respect to Pairs of Parties in Successive Elections*: if party *A*'s vote share increases relative to party *B*'s vote share from one election to the next (while assembly size remains constant), *A*'s seat share relative to *B*'s should not decrease.

Divisor formulas always satisfy this condition, but quota formulas do not.

(M5) *Monotonicity with Respect to District Magnitude (or Assembly Size)*: for a fixed profile of party vote shares, no party should lose (gain) seats if the number of seats available is increased (decreased).

Divisor formulas always satisfy this condition, but quota formulas do not (as was illustrated with respect to LR-H in section 3.4).

(M6) *Majoritarian Constraints*: a party that receives a majority of the vote should be awarded a majority (or at least half) of the seats; conversely, a party that receives a minority of votes should not be awarded a majority of the seats.

No apportionment formula always satisfies either constraint, though some national PR systems write such constraints into their electoral laws, overriding the apportionment formula if necessary.

Finally, we identify two conditions that do not pertain to monotonicity but seem to be appealing conditions for proportional representation.

(PR1) *Staying in Quota*: a party should be awarded a number of seats that is equal to its ideal quota of seats rounded either up or down to the nearest whole number.

The quota formula LR-H by design satisfies this condition, but other quota formulas and all divisor formulas can fail to do so. In particular, D'Hondt may give large parties more seats than their ideal quotas rounded up (as was illustrated in section 3.4). It follows that no apportionment formula both stays in quota and meets monotonicity conditions M4 and M5.

(P2R) *Encouraging Party Fusion/Discouraging Party Schism*: if two parties fuse into a single party, all else remaining constant, the fused party should win at least as many seats as were collectively won by the two component parties; and if one party divides into two parties, all else remaining constant, the separate parties should collectively win no more seats than the united party did.

Since the D'Hondt apportionment formula tends to favor larger parties, it is not surprising that it meets this condition; what may be surprising is that it is the *only* formula that meets this condition. In particular, under LR-H and Sainte-Laguë (modified or not) fusion may cost parties seats and a schism may gain them seats.

As discussed in Module 6, Plurality Rule and thus FPTP at the district level can produce unexpected and unfortunate outcomes, including failure to elect a Condorcet winner; and FPTP can produce similar outcomes at the national level, particularly extreme disproportionality between seats and votes and election inversions. PR elections likewise can produce (arguably) unexpected or unfortunate outcomes. Here are a couple of examples.

PR, like Plurality Rule, takes account of first preferences (with respect to parties rather than candidates) only. Imagine a society divided into three or more somewhat hostile ethnic, language, religious, or other groups, none of which constitutes a majority of the population and each of which has formed its own political party which is the most preferred party of almost all group members. List-PR is commonly recommended for such a 'plural society'. However, suppose that an encompassing (multi-ethnic, multi-lingual, and/or secular) 'alliance party' forms, which is the second preference of almost all voters but the first preference of very few. In this event, the 'alliance' party is preferred by large majorities to each other party (and therefore is the Condorcet winner) but likely wins few if any seats in parliament under list-PR (or STV).[19]

High Legal Thresholds. A high legal threshold under PR can produce an extreme discontinuity between seats and votes. For example, given a 5% threshold and assembly size of about 600 (as in Germany), a party that gets 4.99% of the votes wins no seats, whereas a party that gets 5.01% wins about 30 seats. Moreover if there are a great many such small parties, a large proportion of electorate gains no representation and the larger parties win very disproportionate shares of seats.[20] If a party is predicted to win less than 5% of the votes, some of its normal supporters may defect and vote 'strategically' for their more preferred larger party (in the manner discussed in the following section). On the other hand, if a major and minor party form a tacit electoral alliance (such as the CDU+FDP or SPD+Greens in Germany), the major party may urge some of its voters to vote 'strategically' for its minor party ally so as to help it meet the 5% threshold. In the absence of such strategic maneuvers, a PR system with a legal threshold can produce an election inversions in the manner of the U.S. Electoral College.[21]

3.7 Bibliographical Notes and Further Readings

Rae (1967), Gallagher (1992), Amy (2001), Farrell (2001), Gallagher and Mitchell (2005) and many other works provide detailed descriptions of many electoral rules. Chapters in Grofman et al. (1999) describe the operation SNTV in east Asian countries. Chapters in Bowler and Grofman (2000) discuss the operation of STV in several countries. Chapters in Shugart and Wattenberg (2001) discuss the kind of mixed system pioneered in postwar Germany and variants since adopted in a number of other countries. Tideman (2015) provides a comprehensive description of multi-winner voting rules, including other variants of STV noted in footnote 5. The principle of 'free association' was endorsed by Mill (1861) and has been elaborated by Sugden (1984). Balinski and Young (1982)

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[19] Of course, such a party probably would do no better under FPTP (but probably would do better under Approval Voting or some kind Borda Rule).

[20] Pushing this line of thinking to the limit, only one party might surpass the threshold and win all the seats. Pushing it beyond the limit, no party at all might surpass the threshold.

[21] In the 2013 German federal election, voters in effect were choosing between two prospective coalition governments: the incumbent center-right CDU+FDP coalition and the opposition center-left SPD+Green coalition. The former won a greater vote share than the latter (indeed, than the latter plus the Left Party) but could not form a government (indeed, it won fewer seats than a hypothetical SPD+Green+Left coalition) because the FDP, unlike the Greens and Left, fell below the 5% threshold.

provide a lucid analysis of apportionment methods in the context of apportioning seats in the U.S. House of Representatives, and they present a general mathematical theory of apportionment. Pukelsheim (2014) provides a mathematically oriented treatment of apportionment methods in the context of European PR electoral systems. Rae et al. (1971) introduced expressions for the thresholds of representation and exclusion, which Lijphart and Gibberd (1977) refined and extended. Benoit (2000) assesses the effects of different apportionment formulas and district magnitudes. Miller (2015) establishes that list-PR systems can fail to meet condition M2.

4. Strategic Effects of Electoral Rules

In Module 6, it was noted that no voting rule for electing a single candidate can be *strategyproof* if there are more than two candidates in the field. The same is true of electoral rules for electing several candidates in MMDs. Here we present a general but informal way of thinking about, and analyzing the effects of, strategic voting calculations in both MMDs and SMDs. We assume that voters want to use their votes to influence the outcome of elections in a way consistent with their preferences and not merely to express their preferences regarding candidates or parties (as they might in a public opinion poll). More specifically, we assume that they want to influence the outcome of the *present* election and not, for example, to influence which candidates or parties may enter the field in subsequent elections (which long-term goals might justify other choices in the present election).

4.1 Strategic Voting by Individuals

We first focus on strategic voting under electoral rules using candidate-oriented ballots in MMDs. Let us assume that voters in a district initially intend to cast *sincere ballots*. If an ordinal ballot is used, a sincere ballot ranks the candidates on the ballot exactly in accord with the voter's preference ordering. If a nominal ballot is used, a sincere voter votes for candidate A only if the voter also votes for all candidates he or she prefers to A.

Consider an election in an MMD with magnitude M. Let us suppose that, prior to the election, voters form (on the basis of the historical partisan lean of their district, early pre-election polls, news stories, or other information sources) broadly shared expectations concerning the relative electoral strength of the candidates, i.e., their relative popularity in the electorate and, more particularly, the order in which the candidates will finish in the election (in terms of ordinary votes, approval votes, first preference votes, etc., depending on the electoral rule in use). In the absence of some such information, voters cannot vote strategically.

Suppose that there are n candidates $C_1, C_2, \ldots, C_m, C_{m+1}, \ldots, C_n$, where $n > M$ and the subscripts indicate the relative perceived strengths of the candidates, so that C_1 is the strongest and C_n the weakest. *Leading* candidates are very likely to win, *trailing* candidates are very likely to lose, and *competitive* candidates have uncertain prospects. Clearly, if there are any competitive candidates, there must be at least two. Neither the prospective margins of victory of the leading candidates nor the prospective margins of defeat of the trailing candidates are terribly important. What is important is the relative standing of the competitive candidates, some of whom will win and others of whom will lose. Rather typically there are just two competitive candidates C_m and C_{m+1} and what is uncertain is which one will win a seat and which will lose.[22]

To keep things simple, let us focus on this most typical case. This expectation will evidently cause voters to reconsider, in a strategic fashion, how they should vote and, in particular, will induce some voters to cast insincere ballots. Voters will be induced to change their voting intentions based on strategic calculation only if both of the following conditions hold: first, they have a clear preference between the competitive candidates and, second, their sincere ballots do not already reflect this preference. We call such voters 'concerned'.

Under SNTV, concerned voters who originally intended to vote either for a trailing candidate or for a leading candidate with a comfortable cushion of support may be induced to vote instead for their preferred competitive candidate (since otherwise their votes cannot affect the election outcome and are 'wasted'). Under cumulative voting, concerned voters who originally intended to spread their votes over a number of candidates or to plump them on a trailing or leading candidate may be induced to plump them on their preferred competitive candidate. Under approval voting, concerned voters who originally intended to vote for both or neither of the competitive candidates may be induced to discriminate between them, voting for the preferred one and not the other. Under Plurality At-Large Rule, voters who initially did not intend to vote for either competitive candidate may be induced to switch one of their votes to their more preferred competitive candidate. Under STV, some supporters of a leading candidates who appear to have surplus support given sincere voting may have an incentive to move a preferred competitive candidate to the top of their ballots in order to prevent that candidate from being eliminated for lack of first preference support. However, strategic calculations under STV are so complex that voters may choose to vote sincerely.

In general, such strategic adjustments in voting intentions have two effects on the strength of non-competitive candidates. First, leading candidates (especially those far in the lead) are likely lose some of their prior support, which migrates to competitive candidates. Second, trailing candidates are likely to lose some or all of what little support they originally had, which also migrates to competitive candidates. In general, voters have an incentive to redirect their votes to 'where the action is', i.e., to competitive candidates.

Now suppose a new round of pre-election polls, or other new information pertaining to voting intentions, reflecting these strategic adjustments in intended votes becomes available to voters. Continuing to simplify matters, let us suppose that no new information comes to light that reflects on the merits of the candidates. That is, we suppose that sincere preferences are unchanged since the first poll and that any changes in the standing of the candidates reflect only strategic adjustments in voting intentions. In light of the new polls, voters make can further strategic adjustments in their voting intentions, and so forth through several rounds of polls and adjustments.

We now observe that the first strategic effect is *self-limiting*: if strategic adjustments actually threaten candidate *C*'s leading status, some voters will be induced to redirect their votes back to *C*.

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[22] However, three or more competitive candidates may be clustered in a near (expected) tie at the borderline between leading and trailing candidates. On the other hand, there may sometimes be a substantial gap in strength between C_m and C_{m+1}, so that all candidates are either leading or trailing and the outcome of the election is pretty much a foregone conclusion.

On the other hand, the second effect is *self-reinforcing*: as voters desert a trailing candidate C, C's trailing status becomes even more pronounced and evident, encouraging further desertions, so that C's trailing status becomes still more pronounced and evident.

Thus, after several rounds of polls and strategic adjustments, we can expect an outcome that looks more or less like the following: candidates C_1 through C_{m+1} receive substantial support in the election and M of them are elected (most likely all but C_m or C_{m+1}), while candidates C_{m+2} through C_n receive very little support. This is worth stating as a formal proposition:

(P1) *An election in a district of magnitude M typically ends up with M+1 'serious' candidates*.

'Serious' candidates are leading and competitive candidates; of course, additional (trailing) candidates are on the ballot and receive a few votes. In addition, given MMDs there is some tendency for the vote support for these 'serious' candidates to be more evenly distributed than their support in sincere preferences, because some voters who most prefer very strong candidates feel especially free to desert them in favor of the relatively preferred competitive candidates (but not to the extent that such candidates fail to be elected).

Party-list PR systems largely eliminate such strategic considerations in voting, but with several exceptions. First and as already noted, a legal threshold creates incentives for strategic voting. Supporters of minor parties expected to fall below the threshold have an incentive to vote for their most preferred party expected to surpass the threshold. Conversely, supporters of a major party (that will clearly meet the threshold) may have some incentive to vote for a minor party that is a preferred coalition partner if it is at risk of falling below the threshold. Second, if list PR is applied in small magnitude districts (though it rarely is), the previous argument pertaining leading, competitive, and trailing candidates applies to parties, as discussed just below. Third, even in large magnitude MMDs, the $M+1$ limit on serious parties induced by strategic voting presumably still holds in theory (but lower limits typically apply in practice).

4.2 Strategic Entry and Exit by Candidates and Parties

Except for the remarks immediately above, we have to this point not assumed that candidates share political party affiliations — either each candidate represents a different political party or the election is non-partisan. However, given partisan elections in MMDs, one or several parties may have enough electoral support that they can expect to elect more than one candidate. Recall that, under a quasi-proportional electoral formula, a political party that commands the loyalty of some number of voters in an MMD can expect to elect zero, one, or several of its candidates, depending on how many (Droop) quotas its electoral support contains. On this basis we may distinguish among leading, competitive, and trailing political parties as well as candidates.

We saw earlier that, in order to realize these expectations, leading parties must make strategic calculations, dependent on the electoral rule, with respect to how many candidates they should nominate and how they should urge or instruct their supporters to vote.

On the other hand, trailing parties have a clear incentive to *collaborate* (or engage in *fusion*) by nominating a common candidate so as to (try to) pool their electoral support into a bloc that approaches quota size and may allow them to elect the candidate. Likewise a competitive party can collaborate with a trailing one to gain a full quota of support and be assured of electing at least one candidate. Even a leading party may have an incentive to collaborate with another party in order to be able to elect more candidates. Such collaboration could extend from a temporary expedient in which one candidate and/or party makes a strategic exit from the present election and endorses another candidate and/or party to a full and permanent merger of the parties.

Presumably, such collaboration can profitably occur only among parties that are ideologically proximate or, in any case, not entirely opposed in their policy goals. Suppose there are three parties in an MMD with 100,000 voters in which two candidates are to be elected (so the Droop quota is 33,334) by SNTV. Suppose the Left Party is supported by 24,000 voters, the Center Party by 56,000, and the Right Party by 20,000. Only *C* has a full quota of support and even *C* can be certain to elect only one candidate. In the absence of inter-party collaboration, however, *C* will elect both candidates (provided *C* can sufficiently coordinate the votes of its supporters) and *L* and *R* will elect none. *L* and *R* could take the second seat away from *C* if they pooled their support but — given that they represent opposite ideological extremes — it is unlikely that they could agree on a common candidate that they would both prefer to a *C* candidate.

In general, we expect there to be just $M+1$ 'serious' (leading or competitive) candidates under candidate-oriented systems and therefore also expect no more than $M+1$ parties (and probably fewer) if M is at all large, since several parties are likely to elect more than one candidate. At the same time, we would expect no fewer than two serious parties, because there should typically be a competitive contest for the last (or only) available seat in the district.

In contrast, given party-oriented list PR, strategic incentives for parties are greatly reduced — but not totally eliminated — because list PR allows very large magnitude (including nationwide) districts, which in turn make the quota very small, with the result that many parties — even (depending on any legal threshold) quite small ones — can elect at least a few candidates on their own, with the result that almost all parties with significant support in the electorate are leading. Moreover, given large or nationwide MMDs, the seat-winning capacities of these parties depend on their electoral support in an 'almost continuous' fashion, instead of in the conspicuously 'stepwise' fashion that results when a few large quota thresholds are surpassed. In practice, the principal exception to this generalization occurs when a list PR system imposes a relatively high legal threshold for a party to win any seats. The example of Germany, and the resulting strategic incentives for voters and parties, have already been noted. In theory, the exception also applies when list PR is used in small MMDs, but this rarely occurs in practice.

Finally, since the D'Hondt apportionment formula somewhat favors larger parties, two small parties may be slightly rewarded in terms of seats benefit by combining into a single larger party. (Thus there is an incentive for *apparentement* under D'Hondt but not Sainte-Laguë or LR-H.) Conversely, a party is likely to be slightly penalized in terms of seats if it splits into two smaller parties. Thus D'Hondt, unlike Sainte-Laguë or LR-H, puts a modest brake on party system fragmentation, though not nearly as powerfully as FPTP.

4.3 Strategic Effects in SMDs

We now consider these issues in the special case of SMDs with Plurality Rule, i.e., FPTP. With $M = 1$, the Droop quota is a simple majority, so candidates C_1 and C_2 typically are both competitive and all other candidates are trailing. Of course, if there is something like a three-way (or more extensive) tie for first place, more than two candidates are competitive. Conversely, if C_1 has a substantial lead over C_2, C_1 has leading status and C_2 through C_n are trailing.

With respect to strategic adjustments in voting intentions, the effect on leading candidates cannot occur because either there are no leading candidates or, if C_1 is leading, there are no competitive candidates, so only the self-reinforcing strategic effect on trailing candidates occurs. Duverger's Law at the district level follows as a corollary of proposition P1: an SMD election typically produces just two 'serious' candidates nominated by two 'serious' parties — though additional candidates and parties may be on the ballot and receive a few votes (though probably having greater support with respect to sincere voter preferences).

However, in this case, there is no tendency for the electoral support for these 'serious' candidates to be more evenly distributed than their support in sincere preferences, because — given that only one candidate can be elected — there is no reason for supporters of the strongest candidate migrate elsewhere.[23]

Under particular circumstances, 'non-Duvergerian' outcomes involving three or more candidates with significant support in the final vote may occur. First, C_1 may be so far in the lead with respect to sincere preferences that there are no competitive candidates to whom supporters of trailing candidates may be induced to migrate. Second, C_1, C_2, and C_3 (and possibly additional candidates) may be essentially tied with one another, so it is not clear how votes should be redirected to be more effective (or which candidate should make a strategic exit). Third, C_1 may have a modest lead over C_2 and C_3 (and perhaps additional candidates) who are essentially tied with one another, so even though supporters of these trailing candidates may have an incentive to coordinate their votes on one of them in order to defeat C_1, it isn't clear which candidate they should coordinate on (or which candidate should make a strategic exit). Fourth, many supporters of trailing candidates may be essentially indifferent between C_1 and C_2 and so have no incentive to redirect their votes to either of them. Fifth, if C_3 is trailing but has considerable support, its supporters may remain loyal on the off chance that C_3 may yet win, especially given a very close race between C_1 and C_2 since then a candidate can win with barely one third of the vote. Finally, the ideological configuration among candidates, e.g., a strong centrist candidate who is Condorcet winner but not a majority winner bracketed by somewhat weaker leftist and rightist candidates, may induce equilibrium in the manner described earlier.[24]

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[23] However, as discussed in Module 10, competition between the two candidates or parties may blunt differences in ideology or policy between them and thereby tend to equalize their support in sincere preferences.

[24] Recall from Module 6 that a majority winner is a candidate who is the first preference of a majority of voters while a Condorcet winner can beat each other candidate in a pairwise majority vote.

4.4 Bibliographical Notes and Further Readings

The discussion in this section is in large part inspired by the work of Cox (1997, especially Chapter 5), who in particular developed the '*M*+1 rule', which generalized findings of Reed (1990) pertaining to SNTV in Japan.

5. Votes and Seats at the National Level

Parliamentary elections have the direct effect (if voters are voting for parties) or the indirect effect (if voters are voting for candidates affiliated with parties) of allocating seats to political parties. Thus, a U.K. general election (indirectly) allocates seats in the House of Commons among the Conservative, Labour, and other parties, and a U.S. congressional election (indirectly) allocates seats in the House of Representatives between the Democratic and Republican (and possibly other) parties. Even U.S. Presidential elections can be interpreted as (indirectly) allocating electoral votes ('seats' in the Electoral College) to parties. (See Box XX on The U.S. Electoral College.) On the other hand, a party-list PR election directly allocates seats to parties (and indirectly allocates party seats to individual candidates).

5.1 Deviation from Proportionality

One obvious and important characteristic of an election outcome is the degree to which parties win seats in proportion to their vote shares. Given an assembly with S seats to be filled and an election in which V votes are cast for parties (or candidates affiliated with parties), let V_i and S_i be the number of votes cast for, and the number of seats won by, party i, and let v_i and s_i be party i's vote and seat shares (expressed as either a proportion or a percent). Call the difference $s_i - v_i$ party i's *advantage deviation*, which may be positive (for an *advantaged* party) or negative (for a *disadvantaged* party) or possibly zero (for a party that wins precisely its ideal quota). Clearly an election produces a perfectly proportional result if and only every party receives its ideal quota. While perfect proportionality is (almost) never mathematically possible, a high degree of proportionality is achieved if all party deviations are close to zero, while disproportionality results to the extent that some or many are quite large. The two most commonly used *measures of disproportionality* (but often called 'measures of proportionality') of election outcomes are based directly on the profile of party deviations. Note that a measure cannot be based on the *average* of these deviations, because they necessarily add up to zero.

The first measure D_1 adds up the *absolute* deviations (by ignoring the minus signs for disadvantaged parties) and then divides by two:

(D1) *Measure of Dispropotionality* $D_1 = \frac{1}{2}\sum \left| s_i - v_i \right|$.

There are two (related) rationales for dividing by two. First, in the logically extreme case of disproportionality in which one party gets all the seats though it gets no votes and another party gets no seats though it wins all the votes, the absolute deviations add up to 2 (or 200%), so the sum is divided by 2 in order to constrain maximum disproportionality to 1 (or 100%). Second, dividing by 2 is equivalent to finding the sum of the deviations of all advantaged parties. Thus D_1 can be readily interpreted: for example, if in a 250-seat assembly $D_1 = 0.25$ (or 25%), it follows that the advantaged parties collectively have 0.25×250 = 62.5 more seats than perfect proportionality would give them (and the disadvantaged parties collectively have 62.5 fewer seats). But this property of D_1 also means that D_1 is entirely insensitive as to *how* these 62.5 seats are allocated among the advantaged parties (or how the 62.5 seat deficit is allocated among the disadvantaged parties). Suppose that a highly advantaged party were to 'donate' just enough seats to a slightly advantaged party that they become equally (and moderately) advantaged (or that a slightly disadvantaged party were to 'donate' just enough seats to a highly disadvantaged party that they become equally disadvantaged). While we might think that the effect of such 'donations' from more to less advantaged parties would be to move the direction of greater proportionality, they have no effect on D_1. Moreover, D_1 does not distinguish between a lot of small, mathematically unavoidable, and politically trivial deviations and a few large ones that are both avoidable and politically significant. Consider, on the one hand, a highly fragmented election in which five parties have deviations of +2% and the other half of −2% and, on the other hand, a two-party election in which one party has a +10% deviation while the other has a −10% deviation. We might think that the latter result is considerably more disproportional, but D_1 = 10% in both cases.

A second measure of disproportionality D_2, also based the profile of party deviations, addresses both of these problems. It *squares* of each deviation (thereby producing positive values for disadvantaged as well as advantaged parties), adds up these squares, divides by 2, and then takes the square root of the result:

(D2) *Measure of Disproportionality* $D_2 = \left[\frac{1}{2}\sum\left(s_1 - v_i\right)^2\right]^{1/2}$.

D_2 in effect weights each deviation by itself, with the result that the kinds of 'donations' discussed above do reduce its value and that the two-party case noted above has a greater D_2 value than the 10-party case. When there are only two parties $D_1 = D_2$, but otherwise, it is almost always the case that $D_2 < D_1$. However, D_2 may fall short of 1 (or 100%) even in cases that appear to be maximally disproportional.

Note that both D_1 and D_2 are based party deviations, i.e., the absolute differences between seat shares and vote shares, independent of party size. A party that gets 2% of the seats with 12% of the vote counts the same way as a party that gets 35% of the seats with 45%; both receive seat shares 10 percentage points less than their vote shares. But while the second party receives 77.7% of its ideal quota, the first party receives only 16.7% of its ideal quota. (This consideration may put the 10-party versus two-party examples noted above in somewhat different light.) So we might base measures of disproportionality on the *advantage ratio*s s_i/v_i of the parties rather than their advantage deviations $s_i - v_i$. One way to do this is to focus on the difference between the advantage ratio and perfect proportionality — that is, on $s_i/v_i - 1$. Since these differences, like advantage deviations, can be either positive or negative, we can square them (in the manner of D_2); and since this difference can be much greater for small parties than larger ones, we can weight each squared difference by party size. This gives us $v_i(s_i/v_i - 1)^2$, which simplifies to $(s_i - v_i)^2/v_i$. A third measure of disproportionality is simply the sum of these expressions over all parties:

(D3) *Measure of Disproportionaly* $D_3 = \sum\frac{\left(s_i - v_i\right)^2}{v_1}$.

We may note that D_3 may also by characterized as the square of the advantage deviations for each party *relative to party size* (i.e., vote share) and then summed over all parties. Since this expression has no particular upper limit, there is no reason to divide it by 2 (or any other number) or to take the square root.

A fourth measure of disproportionality focuses directly on advantage ratios and is equal to the advantage ratio of the relatively most advantaged party:

(D4) *Measure of Disproportionality* $D_4 = \max(s_i / v_i)$.

That is, D_4 associates the disproportionality of an election outcome with the advantage ratio of the most advantaged party.

Recall that, in the example in Table 13, we found that each of the apportionments (2), (3), and (4) was produced by one of the common apportionment formulas, but we had a hard time judging which apportionment (and perhaps therefore which formula) was 'most proportional'. Can our measures of disproportionality resolve this matter? The values of each measure for the five apportionments (as well as for the ideal quotas) are shown in the bottom half of Table 13. It can be seen that they agree that if parties could be awarded their ideal quotas, disproportionality would be zero. They also agree that apportionments (1) and (5) are more disproportional than the others, except that D_4 evaluates (4) and (5) equally (since both give P_3 one seat and the same maximum advantage ratio). But they disagree with respect to apportionments (2), (3), and (4): D_1 and D_2 deem (4) to be most proportional, D_3 deems (3) to be most proportional, and D_4 deems (2) to be most proportional. This is not happenstance. It can be shown that LR-H by design minimizes the value of D_1 (and also D_2), that Sainte-Laguë by design minimizes D_3, and D'Hondt by design minimizes D_4.[25] Thus we cannot use these measures to resolve the question of which apportionment formula is 'most proportional'.

Nevertheless, these measures can be of use assessing different national election outcomes under a given electoral system or average national elections outcome under different electoral systems. This is so because, as we have seen, many systems use electoral rules that are at best only quasi-proportional (and not based on any explicit apportionment formula) or do not aim for proportionality at all (e.g., FPTP). Moreover, as we have seen, essentially all list-PR electoral systems that do use an explicit apportionment formula have other features that affect the proportionality of national election outcomes. These include: the use of several or many MMDs (perhaps of varying magnitudes and inevitably presenting somewhat varied ratios of votes cast to candidates elected); the absence (or insufficient number) of national adjustment seats to compensate for such problems; and legal thresholds, an explicit bonus for leading parties, and so forth. It is therefore useful to have a way of comparing the disproportionality of election outcomes over systems. For this purpose, it has become fairly standard to use measure D_2 and, given the complexities just mentioned, it is not predetermined that LR-H systems necessarily come out appearing most proportional. Moreover, given these other complexities, D_1 and D_2, both of which are minimized by LR-H, may give rather different assessments of the disproportionality of election outcomes.

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25 Moreover, other apportionment formulas minimize other measures of disproportionality.

Moreover, in so far as election outcomes may be disproportional, these measures do not address important further questions, in particular which parties are advantaged and which are disadvantaged. Typically, but not always, the leading party or parties are advantaged and small parties are disadvantaged (in particular under FPTP and by small MMDs and/or legal thresholds under list PR). Note further that, in a two-party case in which (say) $v_1 = 55\%$ and $v_2 = 45\%$, D_1, D_2, and D_3 do not distinguish between the case in which $s_1 = 65\%$ (a fairly typical FPTP outcome) and the case in which $s_1 = 45\%$ (a massive election inversion), while D_4 deem the latter outcome to be only slightly less proportional.

Finally, we should bear in mind that these measures pertain to the proportionality of particular election *outcomes*, not of electoral systems per se. In particular (and especially in the case of FPTP), disproportionality may vary greatly from election to election even as the electoral system remains constant. In so far as we are concerned with the proportionality of electoral systems, measures of disproportionality should be averaged over a number of elections. More fundamentally, if we want to compare electoral systems, we should do so in terms of institutional features of the systems themselves, not in terms in particular election outcomes. Section 6 takes up this challenge.

5.2 Votes and Seats with Multi-Member Districts

As we have seen, most electoral systems are *districted*, with the result that the translation of votes into seats takes place in (at least) two steps: first, an electoral rule translates votes into seats within each district; second, seats allocated within each district are aggregated across districts into an overall allocation of seats in the national parliament or other assembly. Recall the terminology that classifies parties by their competitive status within a district. Trailing parties have little electoral support and are unlikely to win even one seat in the district. Leading parties have sufficient support that they can confidently expect to win one or more seats. Competitive parties may or may not win a seat. Given a proportional electoral rule, a party with the support of distinctly less than about one quota of the electorate is likely to be trailing, a party with the support of more than one quota is likely to be leading, and a party with the support of about one quota or a bit less is likely to be competitive. Clearly the share of voter support necessary to put a party into one or other category depends on the district magnitude.

Given a large MMD, any list-PR electoral formula can translate party vote shares into seat shares in a manner that reasonably approximates proportionality. For example, a party that receives 27.139% of the vote in an MMD with $M = 38$ has an *ideal quota* of $.27139 \times 38 = 10.313$ seats. Since seats must be awarded in whole numbers, it cannot actually receive 10.313 seats but, regardless of whether it actually receives 10 seats (26.3% of 38 seats) or 11 seats (28.9%), the result is quite close to proportional.[26] And if results within districts are close to proportional, the national results aggregated across districts must be close to proportional also. Indeed, a leading or competitive

[26] As noted in Section 3, the LR-H apportionment formula by construction always gives each party its ideal quota rounded up or down to the nearest whole number. Divisor formulas do not always stay in quota but usually do so. We will in any event speak informally of parties being ‘rounded up’ or ‘rounded down’ in their seat allocations within districts.

party that is slightly penalized by virtue of being 'rounded down' in one district is likely to be slightly rewarded by virtue of being 'rounded up' in another district, so national results with respect to such parties are likely to be more proportional than most district results.[27] This tendency for the 'rounding errors' to roughly balance out in the national seat allocation may be called the *compensation effect*.[28] However, a party that is trailing in (almost) all districts cannot expect to benefit from a compensation effect, as such a party is consistently 'rounded down' to zero seats in (almost) all districts, even though national proportionality would entitle it to several seats in parliament. For example, a party that wins about 1% of the vote in each of 10 MMDs of magnitude 38 is ideally entitled to about 0.38 seats in each district but would probably be "rounded down" to zero seats in all districts and thus would be allocated zero seats nationally, even though national proportionality with an assembly size of 380 would entitle it to 3 or 4 seats.[29]

Within a small (say magnitude 3-6) MMDs, any list-PR or quasi-proportional rule can translate votes for parties into seats in a manner that is only roughly proportional. (Recall the example in Table 13.) For example, a party that receives 27.139% of the vote in an MMD with 5 seats has an ideal quota of $.27139 \times 5 = 1.357$ seats. But it must receive either one (20%) or two (40%) seats, and in either event the result is not very proportional. Again leading and competitive parties are more or less equally likely to be 'rounded up' or 'rounded down', so the compensation effect again implies that the national seat allocation among such parties is likely to be more proportional than most district allocations. But a trailing party winning no more than about 10-15% of the vote in any district may fail to win seats in any district, though such a party would win about its proportional share if large MMDs had been used.

Beyond the question of MMD magnitude is the fact that MMDs (like SMDs) almost certainly are in some degree malapportioned — that is, the total vote in two districts electing the same number of candidates will not be exactly the same, and more generally the total vote across all districts will not be in exact proportion to their magnitudes — though national adjustment seats may mitigate this problem. Furthermore, legal thresholds, applied at either the district or national levels, create (intended) departures from proportionality.

Finally, electoral rules that are as proportional as possible at the district level may produce seriously disproportionate seat allocations at the national level due the *variance effect* if district magnitudes vary greatly and in a way associated with party strength. Consider a country with one

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27 The situation is different if different parties have trailing status in different districts. In 5.3 we consider the case of a minor party with highly sectional support in the SMD context.

28 Of course, if the electoral rule within districts is somewhat biased in favor of larger parties (e.g., D'Hondt) or smaller parties (e.g., SNTV), this bias tends to accumulate when seats are aggregated nationwide.

29 As previously noted, some electoral system have a separate tier of national adjustment seats to secure greater national proportionality than simple aggregation across MMDs produces. But, at the same time, most such systems also impose a national legal threshold that typically prevents small trailing parties from benefitting from such adjustments.

large metropolitan center containing somewhat over half its voters surrounded by a much larger but sparsely populated rural area. Suppose further that the metropolitan area constitutes one large MMD (e.g., $M = 50$), while the surrounding largely rural area is divided into many small MMDs (e.g., M = 2-3), perhaps following county boundaries. Even if the rural area is awarded no more than its proportionate share of seats (though sparsely populated areas often are so favored), the result is that small parties can win seats much more readily in the metropolitan area than in the rural areas. If left-of-center parties dominate in the former and right-of-center parties dominate in the latter, the latter win about their proportionate share of seats in the metropolitan district but the former win few if any seats in the rural districts. The result may be that right-of-center parties win a substantially greater share of seats at the national level than their share of the national vote and left-of-center parties a substantially smaller share.[30]

5.3 Votes and Seats with Single-Member Districts

The most common criticism of FPTP systems is that they often generate highly disproportionate seat allocations at the national level. While this is often blamed on Plurality Rule, we should note that all list-PR formulas, as well as most quasi-proportional formulas, are logically equivalent to Plurality Rule *when applied to SMDs* — that is, they award the first (and only) seat to the largest party. In fact, Plurality Rule is as proportional as possible given that there is only one seat to be allocated. The lack of proportionality of FPTP systems derives from aggregation across SMDs, so the criticism should be directed at the SMD system, not the electoral rule applied within each district.[31]

Given Plurality Rule within each district, at most one party can command a full quota of votes and, whether or not it commands a full quota, the leading party is always 'rounded up' to one seat while all other parties are 'rounded down' to zero seats. But given SMDs, the scope of the compensation effect is quite restricted. At the extreme, if the same party is leading in (almost) all districts, it is (almost) always 'rounded up' while other parties are 'rounded down', so the leading party wins (almost) all seats nationally.

But we have seen that SMD systems are likely to produce just two competitive candidates in each district. If candidates of the same two parties are competitive across all (or almost all) districts, the compensation effect operates (at least approximately) between these two parties but not with respect to any trailing parties, which are (as always) consistently 'rounded down'. Thus 'minor' parties — with the important exception noted in the next subsection of those with a regional basis of support — are severely unrepresented nationally, both with respect to their actual electoral support and even more so with respect to their sincere support in the electorate that has been attenuated by the strategic incentives discussed in 4.3.

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[30] Of course, this effect again can be neutralized by a sufficiently large tier of national adjustment seats, but some PR countries use MMDs of greatly varying magnitudes without such a tier.

[31] On the other hand, variants of Plurality Rule applied to MMDs clearly are not as proportional as possible and accordingly tend to generate even more disproportionate seat allocations that result from both unnecessary disproportionality at the district level and aggregation across districts.

We now consider the FPTP case in more detail, assuming that strategic effects are operating so powerfully that they produce a perfect two-party system — that is, only two parties, P_1 and P_2, contest elections, with the result that $V_1 + V_2 = V$ and $S_1 + S_2 = S$. Let v_1 and s_1 represent the vote and seat shares for P_1, and likewise for P_2. In this context, FPTP is 'proportional' in the very weak sense that the party that wins a majority of votes typically wins a majority of seats. However, the leading party usually wins a greater share of seats than of votes, and this disproportionality increases with its vote share. Moreover, such limited 'proportionality' does not always hold, as the party that wins a majority of the votes may fail to win a majority of seats — that is, an election inversion may occur.

A particular formulation of the votes-seats relationship under two-party FPTP is provided by a famous proposition known as the *cube law*, which proposes that FPTP elections produce a *seat ratio* S_1 / S_2 that is approximately equal to the *vote ratio* V_1 / V_2 *cubed* (raised to the third power)[32] — that is:

The Cube Law: $$\frac{S_1}{S_2} \approx \left(\frac{V_1}{V_2}\right)^3 .$$

The Cube Law implies that the electoral system treats the two parties even-handedly — in particular, they split seats equally if they split votes equally and election inversions do not occur. However, the cube law also implies that the leading party wins a disproportionate share of seats. For example, if P_1 wins 53% of the vote, the vote ratio is 53/47 = 1.128, so the cube law implies that the seat ratio will be about $1.128^3 = 1.434$, which in turn implies that $s_1 \approx 59\%$. Thus, with a relatively small majority of votes P_1 can expect to win a substantial majority of parliamentary seats. The full votes-seat relationship implied by the cube law is the smooth monotonic (i.e., always increasing) 'S-curve' displayed in Figure 1.[33]

Note that, given the cube law, if a party's vote share increases from 50% to 53% (3 percentage points), its expected seat share increases from 50% to about 59% (9 percentage points). This ratio of change in seats to change in votes (given a relatively equal vote split between the two parties) is called the *swing ratio*, and it indicates the *responsiveness* of changes in seat shares to changes in votes shares. The cube law therefore implies a swing ratio of 3 — that is, given a relatively close split in the vote, a party gains (or loses) about 3% in terms of seats for each 1% it

[32] Since the cube law pertains to ratios, votes and seats can be expressed as actual counts, as shown above, or as shares expressed as either percentages or fractions. The Cube Law is sometimes extended to any two parties under FPTP (and not just to the two major parties in a more or less perfect two-party systems).

[33] 'S-curve' here refers to the elongated S-shape of the curve, not to the fact that it predicts seats won. Figure 1 and subsequent figures show a smooth relationship between votes and seats, in effect assuming that the number of both seats and votes is indefinitely large. In reality, if we observed the vertical axis 'under a microscope', we would see it as composed of many small discrete steps (i.e., the number of seats won increases from 0 to 1, 1 to 2, etc.), so the votes-seats 'curve' likewise would increase in discrete steps. (Indeed, observed 'under an electron microscope', the horizontal axis is also composed of discrete values.)

gains (or loses) in terms of votes. As is also shown in Figure 1, the swing ratio is the slope of the line tangent to the center of votes-seats curve.[34]

The question arises as to why the cube law has attracted particular when it appears to be merely a special case in a family of *power laws* of the same general form:

The Power Laws: $$\frac{S_1}{S_2} \approx \left(\frac{V_1}{V_2}\right)^h .$$

The exponent h is the swing ratio, which is no longer fixed at 3 but rather indicates varying degrees of responsiveness of seat shares to vote shares. When $h = 1$, seat shares are proportional to vote shares. As h falls below 1, responsiveness declines, so that the leading party wins a majority of seats but less than its proportionate share. As h approaches zero, seats are (almost) equally divided between the two parties regardless of their vote shares. As h increases above 1, responsiveness increases, so that the leading party wins more than its proportionate share. As h becomes very large, a party with even a tiny advantage in votes wins (almost) all the seats. Votes-seats curves following power laws with a variety of h values are displayed in Figure 2.

The power laws continue to treat the parties even-handedly, but the equation can be further modified to allow for a *bias* coefficient that favors one or other party and thereby can produce election inversions in sufficiently close elections:

The Generalized Power Laws: $$\frac{S_1}{S_2} \approx b \times \left(\frac{V_1}{V_2}\right)^h .$$

The coefficient b represents bias in the votes-seats relationship: if $b = 1$, the relationship is unbiased; as b increases above 1, bias in favor of P_1 increases; as b falls below 1, bias in favor of P_2 increases. Figure 3 displays several votes-seats curves all with $h = 3$ but with varying b. Note that, if the vote split is sufficiently close, $b < 1$ implies that P_1 fails to win a majority of seats with a majority of votes and $b > 1$ implies that P_1 wins a majority of seats with a minority of votes; in either event, an election inversion occurs.

While the cube law worked to very good approximation in mid-twentieth century British elections (and to somewhat less good approximation in many other FPTP elections), the question arises of what if anything is special about the power $h = 3$ (and the coefficient $b = 1$). More generally, questions arise as to why — or under what conditions — actual votes-seats relationships may follow any power law and why they may be biased in favor of one or other party. To address these questions, we must identify the underlying characteristics of the system of districts that determine the nature of the vote-seats relationship under FPTP.

The starting point is to observe that districts need not be — indeed, almost certainly are not — anything like little replicas of the nation as a whole; rather they almost certainly differ from the

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[34] Note that the curve is very close to being a straight line in the vote share range of about $40\% < v_1 < 60\%$. Clearly a swing ratio greater than 1 cannot hold over the full range of vote shares, since that would imply that a party might win more than 100% (or less than 0%) of the seats.

nation as a whole and from each other in all sorts of ways. Being geographically defined, some are northern while others are southern, some are urban while others are rural, some are more prosperous while others less so, and so forth. As a consequence, districts almost certainly vary with respect to their typical levels of support for each of the two political parties. For example, urban and/or heavily working-class districts may give disproportionate support to the more left-of-center party, while rural, small town, or suburban districts may give disproportionate support to the more right-of-center party.

Let d_i indicate the typical *partisan deviation (with respect to P_1)* of district i; specifically d_i is the expected difference between P_1's vote share in district i and in the nation as a whole, i.e., $d_i = v_i - v_1$.[35] Thus a positive d_i means that district i is P_1-leaning, a negative d_i means that district i is P_2-leaning, and a d_i close to zero means the district i is 'marginal' and usually closely contested, likely to be won by whichever party wins at the national level. We will assume that all districts cast the same number of votes and that all votes go to either P_1 or P_2. This implies that district partisan deviations balance out and add up to zero.

While districts have different partisan deviations, national factors specific to particular elections — for example, current economic conditions, the performance of the governing party, attributes of party leaders, the prevalence of 'time for a change' sentiment, etc. — cause vote support for the parties to vary from election to election. Such factors produce 'swings of the pendulum' at the national level, which typically are approximately duplicated in individual districts. Thus if P_1 wins 55% of the national vote, it can be expected to carry a marginal district with $d_i \approx 0\%$ with about 55% of the vote, win a district with $d_i = +20\%$ with 75% of the vote, and lose a district with $d_i = -25\%$ with 30% of the vote.[36] In general, P_1 can be expected to win any district i such that $v_1 + d_i > 50\%$, and to lose any district i such that $v_1 + d_i < 50\%$, while districts with $v_1 + d_i \approx 50\%$ are toss-ups.

Such considerations make clear that the nature of the votes-seats relationship in a two-party FPTP system depends on the *distribution* of districts with respect to their partisan deviations. Table 14A displays seven hypothetical district distributions given a small system of $S = 15$ districts, each with the same number of voters, and ranked form top to bottom in ascending order of their partisan deviations. Table 14B shows the resulting votes-seats relationships but (in contrast to Figures 1-3) they are presented in tabular rather than graphical form and seats are expressed as actual numbers

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35 In the following discussion (and in accordance with ordinary usage), we express these shares in term of percentages rather than fractions.

36 While approximately uniform swings over districts are typically observed from one election to the next, purely local factors (resulting from a government policy that has had a distinctive impact on a district, a local candidate caught in a scandal, etc.) can produce substantial departures from uniformity. Moreover, factors that shape the partisan deviations of districts certainly change over the long (and sometimes not so long) run. The alert reader will notice that the uniform swing assumption leads to logical inconsistency in extreme cases, e.g., given $v_1 = 65\%$, P_1 would win 105% of the vote in district with $d_i = +40\%$. But few districts have such extreme partisan deviations, and parties rarely win more than 60% of the national vote in two-party systems.

rather than percentages. This table presents *cumulative distributions* that show how many seats P_1 wins when it gets 15%, 20%, etc., of the national vote, i.e., for how many districts it is true that $d_i + 15\% > 50\%$, $d_i + 20\% > 50\%$, etc.[37]

If all districts have zero deviations as in column (1), the party that wins a majority of the vote nationwide wins *all* the seats, in the manner of the votes-seats curve in Figure 2 with h indefinitely large.[38] More realistically, if all districts are very close to being marginal as in column (2), the party that wins (even a very small) majority of the national vote wins almost all the seats. At the other extreme, if just one district is marginal and half of the remaining districts lean heavily in favor of P_1 and half in favor of P_2 as in column (3), seats are split almost equally between the two parties regardless of the division of the national vote (within any reasonable range, e.g., $35\% < v_1 < 65\%$), in the manner of the Figure 2 with h approaching zero. If districts are (approximately) *uniformly* spread over the possible range of partisan deviations as in column (4), seat shares are (approximately) proportional to vote shares, in the manner of the straight-line 'curve' in Figure 2 with $h = 1$. The distributions in columns (5) and (6) form 'bell-shaped' patterns, with most districts concentrated near the center of the distributions and fewer towards the extremes, corresponding to more typical votes-seats curves with h greater than 1 but not extremely large. The spread of these bell-shaped distributions is inversely related to the magnitude of h. It can be checked that the distribution in column (6) approximately follows the cube law, while that in column (5) approximates a curve with h substantially greater than 3.

Note that the distributions in columns (1) through (6) are all *symmetric*, in that they are perfectly 'balanced' around district #8 that falls in the middle position — that is, is the *median* district — in each distribution; put otherwise, all districts other than #8 form pairs i and j such that $d_i = -d_j$. This implies that the *median* partisan deviation is the same as the *mean* partisan deviation (which by assumption is always 0%) and that the resulting votes-seats relationships treat the two parties in an evenhanded fashion. But substantially non-symmetric distributions certainly may occur. For example, the distribution in column (7) is *asymmetric* in that P_1's support is concentrated in a relatively few districts in which it overwhelmingly favored, while P_2's support is spread out over more districts in which it is modestly favored. The resulting votes-seats relationship is *biased* in favor of P_2 — in particular, P_2 wins a majority of seats if it wins anything more than 46% of the vote, so the distribution produces election inversions whenever $50\% < v_1 < 54\%$.

Having established that the relationship between votes and seats depends on the distribution of districts by partisan deviations, the next question is what kind of distribution of districts we may expect in practice. The plausible general answer is that districts are most likely distributed in (approximately) the bell-shaped manner suggested by columns (5) and (6) in 14A. The so-called *normal distribution* results when many small independent effects are aggregated to produce an overall effect, as when many social factors combine to determine the partisan deviations of districts.

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37 The reader may wish to review Box XX on Distributions. The charts in Figures 1-3 are also cumulative distributions.

38 If $v_1 + d_i = 50\%$, it is a toss-up which party wins district i, so each party is credited with half the seat.

But this leaves open the further question of how *spread out* such a normal distribution of partisan deviations is likely to be. The *standard deviation* (SD) measures the spread of any distribution. (See Box XX on Central Tendency and Dispersion.) Note that Table 14A shows the SD for each distribution of districts and that smaller SDs are associated with greater swing ratios.

Typically, FPTP electoral systems have hundreds of districts, so district distributions may best be illustrated graphically. It turns out that the votes-seats relationships resulting from districts that are normally distributed with respect to partisan deviations with varying SDs are virtually identical to those implied by power laws with varying values of h, where a larger SD corresponds to a smaller h (but greater than 1). It further turns out that a normal distributions of district deviations with a standard deviation of about 13.7% implies a seats-votes relationship that corresponds to the cube law (i.e., $h = 3$). Thus, if and when the cube law (approximately) 'works', the implication is that district partisan deviations are (approximately) normally distributed with an SD close to 13.7%.

The U.S. is the major nation that comes closest to having a perfect two-party system, and empirical data from the U.S. was used to illustrate concepts defined in Box XX. The figure within the box shows the distribution of congressional districts (from which members of the House of Representatives are elected) by their partisan deviations relative to national presidential vote in 2008. Districts are grouped into deviation categories two percentage points wide. Clearly this real data is messier than the carefully devised hypothetical examples in Table 14A or the smooth curves in Figures 1-3. While the distribution is more or less bell-shaped, it is also somewhat asymmetric — specifically, there are more extreme positive (pro-Democratic) deviations than extreme negative ones. In this respect, the distribution resembles column (7) of Table 14A. The result is the moderate anti-Democratic bias evident in Figure 5, which displays the resulting votes-seats curve, i.e., the cumulative distribution of districts by partisan deviations.[39]

To get a further sense of how and why districts may vary with respect to partisan deviations, the diagrams in Figure 5 are suggestive. Each panel on the left shows an area about half of which is shaded and half unshaded, over which a grid system has been laid. In the top panel, the shaded areas are very coarse and substantially larger than the grid squares; in the middle panel, they are considerably finer and about the same size as the grid squares; in the bottom panel, they are finer still and substantially smaller than the grid squares. Each grid square can be measured in terms of the proportion of its area that is shaded, and the distribution of squares with respect to this measure is shown in the corresponding panel to the right. Given coarse clusters, the grid squares are either mostly shaded or mostly unshaded, producing a *polarized* or *bimodal* distribution. Given intermediate sized clusters, the grid squares may have any mix of shaded and unshaded areas, producing an approximately *uniform* distribution. Given finer clusters, all grid squares contain a mix of shaded and unshaded areas and produce an approximately normal distribution shown to the right. It is evident that if the clusters become even finer, almost all grid squares would have close to an equal mix of shaded and unshaded areas, so the SD of the (approximately normal) distribution would become extremely small.

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[39] Despite the fact the district distribution is only roughly normal, the characteristic S-curve pattern of Figures 1-3 is evident in Figure 15B (though it shows seats increasing with votes in 435 discrete steps).

Now suppose that each left-hand panel represents a geographical area over which voters are evenly spread and that the shaded areas represent clusters of (for example) working class voters who preponderantly support P_1, while the surrounding unshaded areas represent middle class areas which preponderantly support P_2; suppose also that the grid defines equally populated electoral districts (of unusually regular shape). The panels show three different relationships between *cluster size* and *district size*, which can be interpreted in either of two ways.

The first interpretation is suggested by the way that Figure 6 is actually drawn: a system of districts of fixed size overlaid on partisan clusters of varying degrees of coarseness, indicating how coarseness affects the distribution of district partisan deviations and the resulting votes-seats relationship. Consider two hypothetical and maximally extreme substantive examples. If almost all northerners support P_1 and almost all southerners support P_2 (and supposing these are the only two regions and they have equal populations), there are only two huge partisan clusters and almost all districts (except perhaps a few districts that straddle the two regions) support one or other party overwhelmingly. On the other hand, if almost all women support P_1 and almost all men support P_2, partisan clusters — being individual voters, subdividing even households — are minuscule and (almost) all districts are extremely competitive. In both cases, the parties may be closely balanced, winning about 50% of the vote and controlling about half the seats over the long run. But in the first case, the distribution of districts is highly polarized, so even large shifts in votes produce at most very small shifts in seats, while in the second case the distribution of districts has virtually no spread, so very small shifts in the vote produce huge shifts in seats.

In the second interpretation, the panels show partisan clusters of constant size but districts of varying size; thus the top panel shows a small geographic area and a few very small districts while the bottom panel shows a large area with many much larger districts. The implication is that the votes-seats relationship depends, not only on social factors that shape partisan inclinations, but also on a purely institutional factor, namely, the size of districts (and thus overall assembly size). This interpretation reinforces the general expectation that small districts are likely to be less socially diverse internally than large districts and thus more likely to be more firmly supportive or one party or the other. But this also means that there will more diversity across small districts than large ones, so that the smaller the districts, the greater their dispersion in partisan deviation and the lower the swing ratio.[40]

Finally, we may observe that we have implicitly assumed throughout this discussion that district boundaries are exogenously fixed. But it is well known that district boundaries can be drawn with partisan or other electoral considerations in mind. In U.S. politics, this practice has long been referred to as *gerrymandering*, and the term has spread into academic and international political science.[41] If both parties have veto power over the districting process, the likely result is likely that

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[40] This implies, for example, that the leading party should usually control the U.S. Senate (larger districts, higher swing ratio) by larger margins than the U.S. House of Representatives (smaller districts, lower swing ratio). It also implies that, if presidential electors were selected from Congressional Districts (as has been proposed) rather than statewide on a general ticket, the electoral vote less lopsidedly favor the winning candidate.

[41] The general issue of drawing district boundaries is discussed in detail in Module 8.

they draw the boundaries to create a lot of districts that are safe for one or other party and with few marginal districts, somewhat in the manner of column (3) in Table 14A. Districting on a non-partisan basis (e.g., by an independent boundary commission) is likely to create more competitive districts somewhat in the manner of column (5). But if one party unilaterally controls the districting process, it may try to 'pack' as many supporters of the other party voters as possible into a relatively few districts that are extremely safe for the other party, while creating a large majority of districts that it can expect to carry by small to moderate margins in the manner of column (7) in Table 14A (and, in a less extreme fashion, Figure 5B).

5.3 Three-Party Elections under FPTP

The preceding discussion assumed a perfect two-party FPTP system — that is, one in which Duverger's Law operates so powerfully that only two parties win votes and seats. In this event, an FPTP electoral system operates in a generally reasonable manner. The result of every election is that one or other party wins a majority of seats and can form a government; almost always the same party has won a majority of the vote, though inversions do occasionally occur. Moreover, the winning party's seat share is typically exaggerated relative to its vote share so that, even if the vote is quite close, it has a comfortable a comfortable working majority of seats. Of course in practice, additional minor parties always run in elections, at least in some districts, and win at least a small share of votes and may win a few seats. This implies that the leading party may fail to win a majority of votes, though most likely it still wins a majority seats.

We now consider the operation of an FPTP electoral systems if, contrary to the implications Duverger's Law, an electorally significant third party arises in what had previously been a two-party system.

Given a strictly two-party election between P_1 and P_2, it must be that $v_2 = 100\% - v_1$ in every district in every election. Therefore the distribution of party vote shares over districts can be specified by the vote share for P_1 only, as was done in the tables and figures in the previous section. However, given three (or more) parties, the vote share for P_2 in a given district is not determined (though it is constrained by) the vote share for P_1, so we must plot the distribution of district vote shares for P_1 and P_2 independently in a *two-dimensional* graph such as that shown in Figure 6. Since the distribution of seat shares for P_1 and P_2 is subject to the *constraint* that $v_2 \leq 100\% - v_1$, all plotted points lie inside the *election triangle* formed by the two axes representing the vote shares of P_1 and P_2 together with diagonal line representing the relationship $v_2 = 100\% - v_1$, which becomes the hypotenuse of a right triangle formed with the two axes.[42]

In such a triangle, all districts in which P_1 wins a fixed vote share lie the same vertical line; all those in which P_2 wins a fixed vote share lie on the same horizontal line; and all those in which P_3 wins the same fixed vote share lie on the same line parallel to the hypotenuse. Figure 6 shows such a construction for the vote profile (v_1=30%, v_2=60%, v_3=10%). All districts in which P_1 and P_2 have equal vote shares lie on the diagonal line leading from the origin of the graph (where the vote shares of both P_1 and P_2 are 0%) to the midpoint of the hypotenuse (where the vote shares of both P_1 and P_2 are 50%) . All points below this line represent districts in which P_1 outpolls P_2 and all

[42] Such a three-party election triangle sometimes drawn as an equilateral triangle.

points above it represent districts in which P_2 outpolls P_1. Likewise all districts in which the vote shares of P_1 and P_2 are in some fixed ratio lie on a straight line leading from the origin to the appropriate point on the hypotenuse. Note that any district in which P_1 and P_2 together win all the votes is represented by a point that lies on the hypotenuse so, in the special case of a strictly two-party election, all points fall on the hypotenuse, producing the one-dimensional distribution of districts used in the previous section and rendering the present more elaborate graph unnecessary. But, given a third vote-winning party, some or all points fall in the interior of the triangle and, as the vote shares of P_3 increases, the distance of the plotted points from the hypotenuse increases.

As shown in Figure 6, the election triangle can be divided into three *victory regions* demarcated by the bold lines converging at the center of the triangle (at which point each candidate wins one-third of the vote). Furthermore, each such region can be further subdivided into the region in which the winning candidate wins with a majority of votes and the region in which the winning candidate wins with a plurality but not a majority of votes; in Figure 6, these subregions are separated by dashed lines.[43]

The election triangle in Figure 6 is filled with actual data — namely, election results from English constituencies in the 2010 U.K. general election.[44] By counting the number of plotted points in each victory regions, we see that Conservatives won about 56% of the English seats, Labour won about 36%, and Liberals about 8%. Moreover, we see that, despite their distinctly 'third-party' status, Liberal Democrat (P_3) candidates won at least about 10% of the vote in every district and most of them won between about 10% and 25% of the vote, and about two of dozen won a majority of the vote.

Using this analytic equipment, we can consider the effects of a third party arising in what had previously been a two-party system. Initially, given a strictly two-party election, all districts lie on the hypotenuse, along which they are more or less normally distributed with an SD in the range of 12-15%. This distribution may reflect the varying social class composition of the districts, with one major party doing well in preponderantly working class districts and the other in middle class districts. From election to election and following the 'swing of the pendulum' the districts slide more or less uniformly up and down the hypotenuse, though rarely more that about 10 to 15 percentage points.

Now suppose that a more or less centrist third party 'intervenes' in this heretofore two-party competition with appeal is largely unrelated to the existing dimension of competition. The result is that the cloud of points representing districts spreads out somewhat and moves 'south-west' from

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[43] While an election triangle can accommodate more than three vote-winning parties by combining the vote shares of all parties other than P_1 and P_2, but it cannot indicate how vote shares are divided among the other parties and their victory regions cannot be demarcated.

[44] Data from English constituencies only are used because virtually all constituencies in England had essentially three-candidate (Conservative, Labour, Liberal Democrat) contests, while almost all those in Wales, Scotland, and Northern Ireland included strong (sometimes winning) candidates of 'nationalist' parties as well. A handful English constituencies that did not fit the basic three-party pattern are excluded; these include the Speaker's constituency (since by tradition the Speaker is not opposed by major-party candidates) and one constituency won by a fourth-party (Green) candidate. Vote shares are calculated on a strictly three-party basis.

the hypotenuse, but few if any points move into Victory Region 3.

Here we use an election triangle to represent district results from a strictly three-party contest or one in which votes for any additional parties are ignored.) An election triangle can also represent the results of a amulti-party election in which the votes for all parties other than the two major ones are combined.

5.4 Bibliographical Notes and Further Readings

D_1 was proposed by Loosemore and Hanby (1971) for the purpose of assessing the limits of 'distortion' under common electoral rules and for two decades thereafter was adopted as the standard measure of disproportionality for two decades. Gallagher (1991), by way of arguing that different apportionment formulas aim to minimize different notions of disproportionality, demonstrated that LR-H by construction minimizes D_1, noted its questionable properties, proposed D_2 as a superior alternative, and also identified D_3 and D_4 as alternatives measures based on advantage ratios rather than deviations. While Cox and Shugart (1992) show that LR-H also minimizes D_2, D_1 and D_2 may rank election outcomes very differently with respect to their disproportionality, as Renwick (2015) demonstrates with respect to recent U.K. elections. Taagepera and Grofman (2003) identify 19 distinct measures of disproportionality and assess them with respect to 12 criteria; they find that D_1 and D_2 are probably the most satisfactory. The 'compensation effect' was discussed by Powell and Vanberg (2000) and the 'variance effect' by Monroe and Rose (2002). The Cube Law is examined in Kendall and Stuart (1950). Follow-up work includes Butler (1951, pp. 327-333), March (1957-58), Tufte (1973), and King and Browning (1987). The seats-votes relationship under FPTP (in both the two-party and three-party cases), with particular reference to the U.K., is examined in detail by Gudgin and Taylor (1979), from which Figure 6 is taken. Concerning the 2010 U.K. election, see Curtice (2010).

6. Electoral Systems and Party Systems

This section addresses in broadest terms the relationship between electoral systems and party systems. In doing so, it refines Duverger's Law by summarizing important theoretical work by Rein Taagepera (with various collaborators).[45]

6.1 The Duvergerian Agenda

Recall that Duverger's Law states that majoritarian electoral systems create or sustain two-party systems while proportional electoral systems create or sustain multi-party systems. Empirical evidence, exemplified by Tables 1-12 (which of course are only illustrative, not comprehensive) broadly supports Duverger's Law. Indeed, Duverger's (1954) original claim was based in large part on observation of such election results.

Beyond empirical observation, Duverger distinguished between the 'mechanical' and 'psychological' effects of electoral systems to account for his law. In Section 4, we examined both types of effects, without using Durverger's terms. We noted that small parties are consistently 'rounded down' in FPTP system, with the result (unless they have a geographically concentrated basis of support) that they are denied a share seats in the assembly that is anything like proportionate to their votes shares (and may win no seats at all). These *mechanical effects* are built directly into FPTP systems through aggregation over districts, and they operate immediately in each election. In due course, political actors — politicians, office seekers, political activists, and ordinary voters — recognize and take account of these mechanical effects and so are discouraged from supporting or joining minor parties, producing *psychological* (or *strategic*) effects that reinforce the mechanical ones. It may take some time for the various actors to adjust to the system, but in due course majoritarian electoral systems are likely to produce two-party systems. In contrast, proportional systems (especially those with large MMDs or a substantial tier of national adjustment seats and a low legal threshold) generate almost no mechanical effects favoring or penalizing parties of varying sizes, and consequentially almost no psychological or strategic effects either. This allows a fragmented multi-party system to be sustained over time, though it is not evident exactly how many parties will emerge and endure.

Durverger's generalization is sometimes split into two parts: a 'law' associating majoritarian electoral systems with two-party systems and a 'hypothesis' associating proportional electoral systems with multi-party systems. However, the 'law', being more precise, is arguably often wrong, as suggested by Tables 4-6, while the 'hypothesis', being less precise, is rarely contradicted by evidence, as illustrated by Tables 7-12 where all countries have multi-party systems but the number of parties evidently varies greatly.

While Duverger's Law is commonly interpreted as implying that it is the nature of electoral systems that determines the nature of party systems, it is worth noting that causality may flow in the opposite direction (sometimes called 'Duverger's Law Upside Down'). Electoral laws are not givens

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[45] Taagepera is an Estonian-American political scientist with a Ph.D. in physics, not political science, and his quantitative models of electoral systems reflect this disciplinary background.

but have been chosen (and may be changed) by legislators who write electoral laws — that is, by party politicians, likely with anticipated electoral consequences in mind. Certainly when an FPTP electoral system and a two-party system already exist, leaders of the two major parties have a common interest and political power to resist demands for a change to a proportional system. Conversely, when a proportional electoral system and multi-party system already exist, leaders of the largest one or two parties might have an incentive to switch to a majoritarian electoral system but any such efforts will be intensely resisted by leaders of smaller parties.

A thumbnail sketch of the evolution of electoral systems runs as follows. In countries such as Britain in which parliamentary roots extend to the pre-democratic (and pre-party) era, early parliaments were designed to provide representation for towns, townships, counties, or other localities by assigning them to SMDs or, more commonly, small MMDs. As parliamentary representation became elective (though with severely restricted electorates), Plurality Rule or MMD variants thereof were typically used. In the absence of organized political parties, the question of proportionate party representation did not arise. As the electorate expanded somewhat, parties organized and the existing electoral arrangements favored two-party systems. But when the franchise expanded to include most working-class men, some of these parliamentary regimes switched to proportional representation while others maintained their FPTP systems. Given the prospect of the two existing middle-class parties each winning about 25% of the vote and a new working-class (Socialist or Labour) party winning about 50% of the vote, proportional representation was attractive to leaders of the established parties; this was the pattern in many continental European countries. But where the two established parties were able to fuse into a single conservative party (as in Australia and New Zealand), or where one was largely displaced (as the Liberal Party was in Britain), or where the socialist 'threat' was insubstantial (as in Canada and the U.S.), FPTP systems were maintained. In countries with more recently established parliamentary regimes, the various reformist, nationalist, or revolutionary groups whose efforts overthrew the old order, along with various supporters of the old order, typically found it convenient to agree to some kind of proportional electoral system to secure representation for each group. However, the British FPTP system was implanted in most of its former colonies and was generally maintained when they became independent after World War II. Since then several counties have switched from FPTP to some variant of proportional representation when unusual circumstances arose (e.g., regime change in South Africa, citizen agitation plus a referendum in New Zealand). On the other hand, few if any countries have switched from proportional to FPTP systems, though some have experimented with variants of 'reinforced' PR (as described in 3.4).

Whatever the direction of causality, it would be desirable to refine Duverger's Law so as to establish a quantitative relationship between the (numerical) 'degree of party fragmentation' and (numerical) 'degree of proportionality' of an electoral system. This statement identifies two measurement problems that must be solved: how to quantify 'the number of political parties' (a problem noted earlier in 2.2) and how to quantify the 'proportionality' of an electoral system. We consider these two problems in turn.

6.1 The Effective Number of Political Parties

The problem of measuring the 'degree of fragmentation of party systems' — put otherwise, counting political parties — has been resolved by Taagepera and others in a fairly satisfactory way by a measure called the *effective number of political parties*, according to which parties are weighted in the count according to their relative strengths. Such a weighting system implies that the effective number of parties is not a whole number but takes on fractional values from 1 upwards. Since party strength may be assessed terms of either vote shares or seat shares, there are two separate measures: the effective number of *electoral parties* N_v based on vote shares and the effective number of *assembly parties* N_s based on seat shares. If we let p_i be the (vote or seat) share for party i (expressed as a fraction) and let n be the number of parties that win any votes or seats, the formula for the effective number of parties is:

The Effective Number of Parties: $$N_{ef} = (\sum_{i=1}^{n} p_i^{\,2})^{-1}$$.

That is, we first square the fractional share for each party (producing a smaller fraction), then add up the squared shares (producing a sum no greater than 1), and finally take the reciprocal of this sum. It can be checked that, if all parties have equal shares, N_{ef} is equal to the actual number but, in so far as the parties have unequal support, the effective number is less than the actual number.[46] A practical advantage of this measure is that it can be quite accurately be calculated on the basis of incomplete data such as that presented in Tables 1-12 in which the reported vote or seat shares of all 'other minor parties' and/or 'independents' are combined into a single category, since very small shares have essentially no effect on the overall effective number.

Table 15 shows the application of this formula to the election results in Table 6 to calculate the effective number of electoral and assembly parties in the United Kingdom following the 2010 election. This suggests that the U.K. in 2010 had about three and a half electoral parties and two-and-a-half assembly parties. (It is typical that the latter number is smaller than the first, especially in FPTP systems.) Of course, even while the electoral system remains constant, the effective number of parties (like measures of disproportionality) may vary considerably as party fortunes wax and wane from one election to the next and may be averaged over a period time.

6.2 Proportionality of Electoral Systems

With respect to quantifying the 'degree of proportionality' of electoral systems, the first point to observe is that the measures of (dis)proportionality discussed earlier pertain to particular *election results*, not to *electoral systems* per se. For example, a closely contested two-party FPTP election may produce highly proportional results, but this does not imply that FPTP is a highly proportional electoral system.

Since small MMDs (and certainly SMDs) admit less proportionality than large MMDs, it has been standard practice for many years to take (average) district magnitude M as the best measure of proportionality of electoral systems. But this approach has several problems.

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[46] The effective number of parties may be less than two if one party has a very large share of the votes or seats. While anything that we would recognize as a two-party system has an effective number close to 2, the reverse is not true. For example, a system in which one dominant party has a vote (or seat) share of 0.7 while many small parties divide the remaining share would not be recognized as a two-party system but the effective number of parties is just about 2.

First, while we certainly expect that an increase from $M = 1$ to $M = 10$ (an increase of '9 units of magnitude') may produce a substantial increase in proportionality and party fragmentation, we would also that expect an increase from $M = 101$ to $M = 110$ (also an increase of '9 units') would have much less effect. That is, we would expect proportionality and fragmentation to increase with M but at a decreasing rate.

The second problem is that district magnitude has directly opposite effects on proportionality depending on the nature of the electoral rule used within districts. Given a proportional rule, proportionality *increases* with magnitude but, given a majoritarian rule, proportionality *decreases* with magnitude. This objection is more of theoretical than practical relevance, because majoritarian electoral rules are almost never employed in national (or other large-scale) MMD electoral systems (except perhaps in very small MMDs).[47] As a consequence, empirical studies do show a clear positive relationship between proportionality and district magnitude. However, a theoretically complete classification of electoral systems in terms of degree of proportionality requires that the magnitude dimension be 'unfolded' at its minimum point of $M = 1$, as depicted in Figure 7, to take account of the opposite effects of the two types of electoral rules. For a given assembly size, the most proportional electoral system has a single nationwide district in conjunction with a proportional electoral rule, while the least proportional electoral system also uses a single nationwide district in conjunction with a majoritarian electoral rule — in particular, Party-List Plurality Rule, under which the party with the most votes wins all the seats.

The third problem is that overall assembly size, in addition to district magnitude, surely matters for proportionality. For example, an electoral system with $M = S = 10$ and a given proportional electoral rule clearly has less potential for proportionality than one with $M = 10$ and $S = 250$; in the latter, small parties that fail to win seats in one district may win seats in other districts. (Of course, $M = S = 250$ would be still more favorable to small parties.) On the other hand (and consistent with the previous point), if the Party-List Plurality Rule is used, the system with $M = S = 10$ has no potential for proportionality, as the leading party wins all 10 seats, whereas in the system with $M = 10$ and $S = 250$ at least two parties (and perhaps more) are likely to lead in some districts and thus win some seats.

Taagepera has proposed that the appropriate institutional measure of an electoral system to indicate its potential for proportionality is its *seat product* (SP). Typically SP = $M \times S$ — hence its name. However, to take account of the (mostly only theoretical) possibility that majoritarian electoral rules may be used in MMDs, the M term needs to carry an exponent F (so that SP = $M^F \times S$), where $F = +1$ if the electoral rule is proportional and $F = -1$ if it is majoritarian. In the latter case, the name might better be 'seat-quotient', i.e., S/M, which, it may be noted, is simply the number of districts. At the lower extreme in Figure 7 with $M = S$ and a majoritarian rule, the seat product is $S/S = 1$. At the upper extreme with $M = S$ and a proportional rule, SP = $S \times S = S^2$. At the midpoint where $M = 1$ and majoritarian and proportional rules overlap, SP = S.

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47 As previously noted, the principal exception is the Party-List Plurality rule used to elect U.S. presidential electors.

6.3 The Seat Product Model

Taagepera has recently proposed a number of quantitative relationships between the seat product and characteristics of party systems.[48] Taagepera had earlier proposed that assembly size S tends to equal the cube root of the size of the population P represented by the assembly. He derives this quantitative relationship on the supposition that constitution-makers, in choosing S, implicitly seek to minimize both the total number of communications channels between members and constituents (to facilitate the representative character of the assembly) and also among members themselves (to facilitate the legislative work of the assembly). The former declines with the inverse of S; the latter increases approximately with S^2. Balancing these two considerations, it turns out that the total communications burden is minimized when S is equal to approximately the cube root of population.

(T1) *In a jurisdiction with population P, expected assembly size is* $S \approx P^{1/3}$.

Like the seat product laws set forth later, this 'cube root law of assembly size' is broadly supported by empirical data.[49]

Beyond T1, Taagepera's quantitative relationships pertain to the seat product of an electoral system and expected characteristics of a party system — in particular, the number of seat-winning parties, the effective number of electoral and assembly parties, and the vote and seat shares of the leading party. The seat product model produces expectations that are consistent with Duverger's Law as traditionally stated but are more precise by virtue of being strictly quantitative. In contrast to typical social science results based on statistical analysis of data, these quantitative relationships are logically derived prior to examination of relevant data. Taagepera's general mode of investigation is to identify what must logically be true at the extremes of a relationship and then to form an overall expectation, typically by taking the geometric mean of the extremes. The resulting relationships are equations that can be manipulated and connected to one another by ordinary algebra and thereby can all be expressed in terms of the seat product.

The equations aim to predict expected values of the characteristics of party systems. The next necessary step, of course, is to test these equations against data from many elections under many electoral systems. Taagepera regards his equations to be broadly supported if about half the cases have higher values than predicted and about half lower, and as reasonably precise if few cases have values that are more than twice or less than half the predicted value. Within that range, other factors — including further details pertaining to the electoral system as well as external factors such as political history, ethnic and other social cleavages, and other political institutions (e.g., president-ialism, federalism, primary elections) — have room to influence party system characteristics.

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[48] Since these quantitative relationships are expressed by equations involving exponents, the reader may wish to review Box XX.

[49] According to the cube root law, the U.K. House of Commons is considerably 'oversized' and the U.S. House of Representatives is considerably 'undersized'. (During the nineteenth century U.S. House size was regularly increased while population increased rapidly and followed the cube root law very closely; however, in 1913 House size was frozen at 435 while population has continued to increase rapidly.)

Taagepera's first step is to consider the *expected number of seat-winning parties* n_{S0} in a single district of magnitude M when a proportional electoral rule is in use. (Of course, if $M = 1$, $n_{S0} = 1$ regardless of the electoral rule.) At one extreme, one party might win all the seats; at the other extreme, each seat might be won by won by a different party. Thus the logical extremes are $1 \leq n_{s0} \leq M$. In the absence of any other information, we would guess that n_{S0} is some average of these extremes, but the further question is what type of average? For various reasons, the *geometric mean* provides the best estimate.[50] This gives the first of Taagepera's quantitative relationships pertaining to party systems:

(T2) *In a district of magnitude M, the expected number of seat-winning parties* $n_{S0} = M^{1/2}$.

From this Taagepera derives a function of the seat product for the *expected number of seat-winning parties* N_{S0} in the assembly as a whole. Given T2, the expected lower extreme for N_{S0} is $M^{1/2}$ (if the same parties win seats in every district). Given a proportional electoral rule, the most favorable condition for many parties to win seats in the assembly is if all members are elected a single nationwide district (i.e., $M = S$), in which case, again by T2, N_{S0} is $S^{1/2}$. Thus the expected logical extremes are $M^{1/2} \leq N_{S0} \leq S^{1/2}$. Taking the geometric mean of these bounds gives $N_{S0} = (M^{1/2} \times S^{1/2})^{1/2} = (M \times S)^{1/4}$.

While the argument above assumes a proportional electoral rule and therefore covers only the upper half of the classification displayed Figure 7 (but including the FPTP case), any MMD system with a reliably majoritarian rule (in particular, Party List Plurality Rule) is effectively equivalent to an FPTP system with k districts, where $k = S/M$. For example, given Party List Plurality Rule, a system with $S = 100$ and $M = 5$ operates in the same way as an FPTP system with $S = 20$ except that the number of seats won by each party is scaled up by a factor of 5; moreover, the FPTP system and its scaled-up MMD counterpart have the same seat products, $M^F \times S = 1 \times k = 20$ in the first case and $M^F \times S = 5^{-1} \times 100 = 20$ in the second.

Thus, for both types of electoral rules, we have the next of Taagepera's quantitative relationship:

(T3) *The expected number of seat-winning parties in the assembly* $N_{S0} = (\mathrm{SP})^{1/4}$.

Taagepera next derives a function of the seat product for the *expected share of seats s_1 won by the largest party*, where s_1 is expressed as a fraction. At the lower extreme, the largest party logically cannot win fewer seats than the average share of seats won by all seat-winning parties, so $1/N_{S0} \leq s_1$; at the upper extreme, certainly $s_1 \leq 1$. Thus the logical bounds are $1/N_{S0} \leq s_1 \leq 1$. The geometric mean of these bounds gives us $s_1 = (1/N_{S0} \times 1)^{1/2} = N_{S0}^{-1/2}$. Substituting T3 for N_{S0} gives the next quantitative relationship:

(T4) *The expected share of seats won by the leading party* $s_1 = (\mathrm{SP})^{-1/8}$.

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50 The geometric mean of two (positive) numbers is the square root of their product, in this case $(1 \times M)^{1/2}$. In contrast, the ordinary (arithmetic) mean, i.e., $(M+1)/2$, produces inconsistent expectations, for we can also ask about expected average number of seats won by seat-winning parties. The logical extremes again range from 1 to M, but (unless $M = 1$) it is not possible for $(M+1)/2$ parties to win an average of $(M+1)/2$ seats each, whereas $M^{1/2}$ parties can — indeed must — win an average of $M^{1/2}$ seats each.

Taagepera then derives a function of the seat product for the *expected effective number of assembly parties* N_S. Recall that s_1 contributes most importantly to this quantity and that (given S of typical size) the seat shares of very small seat-winning parties contribute almost nothing. For a given value of s_1, N_S can be as low as $1/s_1$ (if all seat-winning parties win the same number of seats) or almost as high as $1/s_1^2$ (if other seat-winning parties win only one seat each). Taking the geometric mean of the logical bounds gives $N_S = (s_1^{-1} \times s_1^{-2})^{1/2} = s_1^{-3/2}$. However, these extremes are problematic. With respect to the lower extreme, all parties can have equal seat shares only in the event that $s_1 = 1/2$ and $N_{S0} = 2$, or $s_1 = 1/3$ and $N_{S0} = 3$, and so forth, and this is especially unlikely if s_1 is a relatively large fraction (as it usually is). And, if S is of typical magnitude and s_1 is not exceptionally large, the upper extreme implies that the number of seat-winning parties must be very large — clearly larger than the expectation given by expression (2). By further refinements beyond the scope of this summary, Taagepera arrives at the revised expectation that $N_S = s_1^{-4/3}$. Substituting T4 for s_1 gives the next quantitative relationship:

(T5) *The expected effective number of assembly parties* $N_S = (\text{SP})^{1/6}$.

Taagepera has more recently derived functions of the seat product pertaining to electoral parties, despite the fact that features of the electoral system do not directly constrain outcomes with respect to votes in the way that they directly (or 'mechanically') constrain outcomes with respect to seats. However, as we have seen (and as Duverger's Law suggests), features of the electoral system indirectly constrain votes through the kinds of strategic (or 'psychological') effects discussed in Section 4. On this basis, it is possible to derive expectations about electoral parties based on the seat product.

The key is P1 informally deduced in 4.1: an MMD with magnitude M typically results in an election with $M+1$ 'serious' (leading or competitive) candidates receiving substantial vote support. Analogously, we may expect that the number N_{V0} of parties 'seriously' contesting an election — after voters, candidates, and parties have made strategic adjustments — to be one greater than the number of parties that actually win seats, i.e., $N_{V0} = N_{S0} + 1 = (\text{SP})^{1/4} + 1$.[51] Of course, other — perhaps many — 'unserious' minor parties each winning a very small fraction of the vote may run, but these parties have essentially no impact on the effective number of electoral parties. We may further expect that the relationship between the effective number of electoral parties and the actual number of 'serious' electoral parties to be similar to that between the effective and actual numbers of assembly parties. From T3 and T5, it follows that $N_S = N_{S0}^{2/3}$. Thus we also expect that $N_V = N_{V0}^{2/3} = (N_{S0} + 1)^{2/3}$. Substituting T3 for N_{S0} gives the following:

(T6) *The expected effective number of 'serious' electoral parties* $N_V = [(\text{SP})^{1/4} + 1]^{2/3}$.

Finally, Taagepera derives a function of the seat product for the *expected vote share of the leading electoral party* v_1. At the lower extreme, v_1 must be greater than the average vote share of all 'serious' electoral parties, i.e., $1/N_{V0}$; at the upper extreme, certainly $v_1 \leq 1$. Taking the geometric

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51 Note that while N_{S0} designates the actual number of seat-winning parties, N_{V0} does not designate the 'actual' number of vote-winning parties, i.e., those that win any votes, but the number of 'serious' vote-winning parties, i.e., those that actually win seats *or come close to doing so*. The actual number of vote-winning parties depends on detailed rules pertaining to ballot access for parties, independent candidates, write-in votes, and so forth.

mean of these extremes gives $v_1 = (N_{V0})^{-1/2} = (N_{S0} + 1)^{-½}$. Substituting expression T3 for N_{S0} gives the following:

(T7) *The expected vote share of the leading electoral party* $v_1 = [(\mathrm{SP})^{1/4} + 1]^{-1/2}$.

Table 16 shows quantitative predictions pertaining all these aspects of party systems given a (rather typical) assembly size of $S = 250$ and selected district magnitudes for both majoritarian and proportional electoral rules.[52]

In principle, these quantitative relationships apply only to *simple* electoral systems — that is, systems whose institutional features are fully specified by the three parameters *S*, *M*, and *F* that define the seat product. As we have seen, most electoral systems have significant added complexities such as: varied district magnitudes, less than fully proportional (or majoritarian) electoral rules, multiple tiers of districts (perhaps with different electoral rules), national adjustment seats, legal thresholds, and so forth. For systems with varying magnitudes, mean district magnitude can replace *M* provided the variation is not too extreme.[53] The exponent $F = +1$ can be reduced somewhat (e.g., to +0.8) for D'Hondt, STV, SNTV, etc. Multiple tiers do not present a problem in the event the top tier provides an adequate number of national adjustment seats to compensate for disproportionality in lower tiers (but often it does not).

To take a specific example, the German electoral system, despite being 'mixed' as described in 3.5, provides enough adjustment seats that it is in this respect essentially as equivalent to a simple system with $M = S$. More problematic is Germany's 5% national legal threshold — a feature that greatly reduces its proportionality relative to what its seat product would suggest. In some respects, the threshold may be accounted for by considering what district magnitude would by itself entail an approximately similar threshold. At first blush, the answer might seem to be $M = 20$, since 5% of 20 seats is a single seat. This suggests that the 'effective threshold' implied by district magnitude is $100\%/M$, but further thought reveals that this ratio is surely too large — in any case, it clearly does not work for SMDs. A common, though somewhat arbitrary, rule of thumb is that the *effective threshold* entailed by a given district magnitude *M* is approximately $75\%/(M{+}1)$. This suggests that the German national legal threshold of 5% has the same overall effect on the party system as using districts with a magnitude of about 14. In fact, inserting a seat product of 14×650 (the approximate size of the German parliament) in expressions (T4) through (T7) gives reasonably good approximations for the German party system. However, (T3) implies about 10 seat-winning parties — a number that is substantially too large. This is to be expected; if Germany had districts with $M = 14$ rather than a national legal threshold of 5%, small parties winning somewhat less than 5% of the national vote would no doubt fail to win seats in most districts but would likely win seats in some districts. This consideration reinforces the basic point that proportionality depends on *S* as well as *M*, i.e., on the seat product rather than district magnitude only.

When adjustments along these lines are made, empirical work broadly confirms the expectations generated by the seat product laws even when applied to many somewhat complex systems.

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[52] The last column pertains to the additional quantitative relationship T8 introduced in 7.2.

[53] For example, the hypothetical example presented at the end of section 5.2 would be 'too extreme'.

6.4 Bibliographical Notes and Further Readings

Duverger's Law is due to Duverger (1954, especially pp. 216-228) but had antecedents in earlier political science, as documented by Riker (1982); also see Benoit (2006 and 2007). On 'Duverger's Law upside down', see in particular Colomer (2005). On the evolution of electoral systems, see Colomer (2004 and 2007) as well as Carstairs (1980). The 'effective number of political parties' was first proposed by Laakso and Taagepera (1979) and has been widely adopted; their measure is logically related to Rae's (1967) earlier definition of 'party fractionalization'. Rae (1967; 1995), among many others, used district magnitude as an indicator of electoral system proportionality. The cube root law of assembly size was first introduced in Taagepera (1972) and refined in Taagepera and Shugart (1989), Chapter 15. Early versions of quantitative relationships between electoral systems and party systems are found in Taagepera and Shugart (1993) and Taagepera (2001). Taagepera (1999) explicated his 'ignorance-based' method of deriving such relationships. Taagepera (2007) introduced the seat product as a summary indicator of the potential proportionality of electoral systems and also derived T3 through T6. More recently, Shugart and Taagepera (2017) have derived T6 and T7. Taagepera (2007) provides preliminary empirical testing of these propositions, while Shugart and Taagepera (2017) provide much more comprehensive testing. Shugart and Taagepera (2018) summarize much of this work, as does Colomer (2017). Taagepera (1998) summarizes the relationship between district magnitude and effective threshold. Li and Shugart (2016) and Shugart and Taagepera extend the seat product model to cover other complexities in electoral systems (upper tiers and presidentialism).

7. Parliamentary Government Formation

In any system of democratic government, the electoral system functions directly to fill seats in parliament or other representative assembly. But in a *parliamentary* (as opposed to *presidential*) system, the electoral system also functions, directly or indirectly, to select the 'government' (or executive) — that is, a prime minister and cabinet supported by a governing party or coalition of parties. (In a presidential system, the executive is elected separately.)

7.1 Electoral Decisiveness and Government Formation

Under an FPTP electoral system, the assembly reflects the character of the electorate at best only imperfectly; while it provides excellent representation with respect to geography and reasonably good representation with respect to supporters of the major parties, it provides poor representation with respect to supporters of minor parties, and its representative character is uncertain in other respects (e.g., ethnic or gender balance). On the other hand, a list-PR electoral system provides excellent representation with respect supporters of parties of all sizes (except perhaps the very smallest) and is typically more representative in other ways (though much depends on the procedures parties use for selecting lists of candidates).

However, its supporters claim that the redeeming virtue of an FPTP parliamentary system, is that, provided that Duverger's Law is operating to good approximation, the electorate effectively chooses the government — that is, FPTP parliamentary elections are almost always *decisive* in that the election itself, rather than negotiation among party leaders after the election, determines what government is formed. Following an election, one or other of the two major parties wins a majority of seats and has a clear mandate to form a government and carry out policies and programs outlined in its election manifesto or platform. At the same time, the runner-up party wins almost all of the remaining seats and forms an official opposition that holds the government to critical account until the next election, at which time the electorate is presented with a simple choice: keep the government in power or toss it out in favor of the opposition.

We have seen that this claim in support of FPTP must be qualified in a number of respects. First, even if Duverger's Law is operating perfectly (so that the two major parties together win all the seats), it is possible that the party that forms the government has won less support in the electorate than the prospective opposition, since election inversions may occur. Second, if Duverger's Law is operating somewhat imperfectly (so that minor parties win more than a trivial share of votes), it is possible or even likely that the party that wins a majority seats is supported by only a plurality, rather than a majority, of the electorate — put otherwise, that the government is formed on the basis of a 'manufactured majority'. Third, if Duverger's Law is operating still more imperfectly (so that minor parties win more than a trivial share of seats), an FPTP election may actually be *indecisive* in that no party wins a clear majority of seats, producing a 'hung parliament'.[54] And, as we saw in 5.4, if Duverger's Law substantially breaks down and minor parties win a considerable share of seats, the operation of an FPTP system may become quite erratic and its principal claimed virtue is thrown into question. In this last event, an FPTP parliament may resemble a PR parliament with respect to party system fragmentation, but with the important difference that seats are likely to be distributed among parties in a way that is very imperfectly related to their electoral support.

In contrast, in a parliament elected by proportional representation seats are reliably distributed in a way that closely reflects their electoral support. Since elections are rarely decisive, seat shares become resources in a post-election bargaining process among party leaders to form a government with majority support in parliament. Since seat shares are (approximately) proportionate to vote shares, it is tempting to conclude that parties have power to influence parliamentary government formation that is proportionate to their share of support in the electorate. However, this conclusion does not follow. Given that party groups act as *blocs* (or 'caucuses' in the language of 2.1) in parliament, coalition possibilities are actually quite limited provided the number of seat-winning parties is relatively small and — to anticipate a major theme in Module 15 — *voting power is not proportional to voting weights*, where voting weights here refer to seat shares.

7.2 Coalition Possibilities in Multi-Party Assemblies

Typically, in parliaments elected by PR, no party controls a majority of seats, and it is

[54] In fact, every U.K. election since World War II has produced either a 'manufactured majority' (including two inversions) or a 'hung parliament' (both terms are distinctively British). Over one-third of U.S. presidential elections have produced manufactured majorities with respect to electoral votes, including four inversions, though there has been no 'hung Electoral College' since 1824.

therefore necessary for two or more parties to enter into a coalition in order to form a government supported by a parliamentary majority.[55] The process of parliamentary government formation therefore constitutes a *simple game* (as discussed in Box 11 and also in the introduction to Module 15), in which the aim is to form a *winning coalition* — that is, one that controls a majority of seats and is therefore able to form a government. More specifically, the process constitutes a *weighted majority game* in which the weights are the seats controlled by each party and a coalition is winning if and only if it controls a majority of seats. To simplify exposition, we shall assume that the total number of seats S is odd, so that the simple game is *strong* and *blocking coalitions* (in this case, complementary coalitions each controlling precisely $S/2$ seats) cannot arise.[56] We shall see that, even though proportional electoral systems may translate the (party) preferences of voters into seat shares in a highly proportional manner, these seat shares do not in turn translate into bargaining power in a similarly proportional manner.

Consider the case in which four parties, labelled in descending order of their seat shares, win the seat shares shown in column (1) of Table 18. While the seat shares of all parties are relatively equal, their bargaining power in terms of potential to form winning coalitions is not at all equal. Rather P_1 has a distinctively stronger position than the other three parties: P_1 can form a winning coalition with any one of them and can be excluded from a winning coalition only if all three other parties enter into a (winning) coalition. Thus the *minimal winning coalitions* — that is, winning coalitions that would become losing in the event of the loss of any member — for this profile of seats shares are $\{P_1,P_2\}$, $\{P_1,P_3\}$, $\{P_1,P_4\}$, and $\{P_2,P_3,P_4\}$.

Now suppose a new election shifts the seat distribution to that shown in column (2). Despite the substantial seat shift (presumably reflecting a corresponding shift in underlying voter preferences) toward P_1 at the expense of P_3 and (especially) P_4, coalition possibilities and bargaining power remain unchanged. Or suppose that P_1 and P_2 both gain seats at the expense of both P_3 and P_4, as shown in column (3). Despite the radical shift in preferences, coalition possibilities and bargaining power still remain unchanged. On the other hand, suppose that the new election instead produces the seat distribution shown in column (4). Though the evident shift in voter preferences is more modest, coalition possibilities and bargaining power have now changed dramatically. Despite gaining seats relative to column (1), P_1 has lost its distinctively powerful position, as it can form winning coalitions only with P_2 and P_3, while P_2 and P_3 can form a winning coalition with each other. Meanwhile, even though it controls more seats than in columns (2) or (3) P_4 has been rendered powerless, as it cannot convert any losing coalition into a winning one by joining it — put otherwise, P_4 belongs to no minimal winning coalition and is therefore a *dummy*.

In fact, given four parliamentary parties, there are only three configurations of coalition possibilities: (a) one party controls a majority of seats and constitutes a one-party (minimal) winning

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55 While *minority governments* sometimes form, they either command additional implicit support short of a formal coalition or aim to win additional support on an ad hoc basis as different issues arise.

56 Surprisingly, quite a number of parliaments and other assemblies have an even number of members. However, given that S is usually quite large, blocking coalitions rarely arise.

coalition, making the other three parties dummies; (b) any pair of three parties is winning and the fourth party is a dummy, as in column (4); and (c) a 'strongly dominant' party can form a winning coalition with any one other party, as in columns (1), (2), and (3). These and other cases may be represented in the following manner.[57]

A weighted majority game may be represented by a profile of *minimal integer weights* (MIWs) w_i, one for each of the n players (in this case, parties) labelled in descending order of their weights (in this case, seats) and where a dummy has an MIW of 0, together with the *quota* q for winning such that, given any pair of complementary coalitions, exactly one has a total MIW equal to or greater than q and is therefore winning. Whatever the distribution of (non-minimal) weights (i.e., seats) among the parties, coalition possibilities are just thus implied by the MIW configuration. In general, a MIW configuration is given by $[q: w_1, w_2, \ldots, w_n]$. Thus, three configurations of coalition possibilities given four parties noted above correspond to the three MIW configurations (a) [1: 1, 0, 0, 0], (b) [2: 1, 1, 1, 0], and (c) [3: 2, 1, 1, 1]. Since in each case, the total MIW of each minimal winning coalitions is equal to q, these are called *homogeneous* weighted majority games.

The number of MIW configurations increases rapidly as the number of (non-dummy) parties increases. While there is only one weighted majority game with four non-dummy parties — namely (c) just above — there are four with five such parties, 14 with six, and 91with seven.[58] However, given a reasonably large assembly, these numbers are still much smaller than the number of distinct seat profiles for the same number of parties.

While weighted majority games proliferate rapidly once the number of non-dummy parties is six or greater, it is possible to present a simplified typology of coalition possibilities stated in terms of the seat shares of the three largest parties (together with several subtypes that refer to seats shares of smaller parties). Let $S_1, S_2, \ldots, S_n$ be the number of seats held by parties $P_1, P_2, \ldots, P_n$ respectively, labelled so that $S_1 > S_2 > \ldots > S_{n-1} > S_n$. (We assume that there are no ties among parties with respect to seats held.)

Type I. There is a *majority party* such that $S_1 > S/2$. Thus P_1 is a one-party winning coalition and thus is the only minimal winning coalition.

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[57] While many seat distributions produce the same configuration of coalition possibilities, they vary with respect to their *fragility* — that is, whether a small shift in seats between elections (as might result from a few by-elections, resignations, deaths, party defections, etc.) may produce a different configuration of coalition possibilities. Note that all profiles (1)-(4) are quite fragile, while the seat profile 42%/21%/19%/18% is *robust* against any small perturbations.

[58] The four weighted majority games with five non-dummy players are [3: 1, 1, 1, 1, 1], [4: 3, 1, 1, 1, 1], [4: 2, 2, 1, 1, 1], and [5: 3, 2, 2, 1, 1]. Given six or more non-dummy players, non-homogeneous weighted majority games appear, such that some minimal winning coalitions have a total MIW greater than q; an example is [5: 2, 2, 2, 1, 1, 1]; note that this game is close to being a simple (unweighted) majority game, in that any majority (4/6) of players is winning but an overall 3/3 tie is resolved by a majority of the privileged players P_1, P_2, and P_3. Moreover, there are strong simple games with six or more players that are not weighted majority games at all, in that it is impossible to assign weights to players such that winning coalitions always have greater total weight than losing coalitions; however, such games clearly are not relevant to parliamentary government formation.

Type II. There is a *strongly dominant party* such that $S_1 < S/2$ and $S_1 + S_3 > S/2$, but $S_2 + S_3 < S/2$. Thus P_1 can form a winning coalition with either P_2 or P_3 but the coalition of P_2 and P_3 is not itself winning.[59] It follows that $S_1 > S/4$, and that both P_2 and P_3 must be members of every winning coalition that excludes P_1, so every such coalition must include at least three parties. Two subtypes can be defined in terms of seat shares of parties smaller than P_3

Subtype IIa. There is a *system-dominant party* such that $S_1 + S_n > S/2$. Thus P_1 can form a winning coalition with *any* other party.[60] The four-party game [3: 2, 1, 1, 1] provides a specific example; the general case is given by $[q : q-1, 1, \ldots, 1]$. It follows that any winning coalition excluding P_1 must include all other parties.

Subtype IIb. There is a *k-dominant party* such that $S_1 + S_k > S/2$ but $S_1 + S_{k+1} < S/2$ for $k \geq 3$. Thus P_1 can form a winning coalition with P_k but no smaller party. A strongly dominant party includes the special case of a k-dominant party with $k = 3$; a system dominant party is the special case with $k = n$.

Type III. There is a *top-three party system* such that $S_2 + S_3 > S/2$. Thus any two of the three largest parties can form a winning coalition. It follows that no coalition excluding two of the largest three parties is winning and that all other parties are dummies. The four-party MIW configuration [3: 1, 1, 1, 0] provides a specific example; the general case is given by [3: 1, 1, 1, 0, . . . , 0].

Type IV. There is a *top-two party system* such that $S_1 + S_2 > S/2$ but $S_1 + S_3 < S/2$. Since $\{P_1, P_2\}$ is the only two-party winning coalition, every other winning coalition must include either P_1 or P_2 and at least two other parties. Type IV subdivides into two mutually exclusive subtypes defined in terms of seat shares of parties smaller than P_3.

Subtype IVa. There is an *asymmetric top-two party system* in which $w_1 > w_2$. Thus P_1 has more opportunities to form winning coalitions excluding P_2 than P_2 has excluding P_1 (for example, [5: 3, 2, 1, 1, 1, 1]).

Subtype IVb: There is a *symmetric top-two party system* in which $w_1 = w_2$. Thus P_1 and P_2 have equal opportunities to form winning coalitions excluding the other (for example, [6: 3, 3, 2, 1, 1, 1]).

Type V. There is an *open party system* such that $S_1 + S_2 < S/2$. This category includes all cases not included in Types I-IV and implies that every winning coalition includes at least three parties. The five-player (simple majority) game [3: 1, 1, 1, 1, 1] provides the only example of an open system with fewer than six parties.[61] A sufficient (but not necessary) condition for an open system is that

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[59] As the name suggests, a more general category of a 'dominant player' exists, but its definition is not straightforward and will not be introduced here.

[60] A slightly more generous definition of system dominance is that P_1 can form a winning coalition with any other *non-dummy* party.

[61] There are five open six-party systems (excluding dummies), though only one of them is homogeneous.

$S_1 < S/4$.

The elements of this typology (including the subtypes) can be ranked terms of their 'degree of openness' as follows: I < IIa < IIb < II < III < IVa < IVb < V.[62] Beyond this, as the number of seat-winning parties increases, open systems themselves can vary greatly in terms of their 'degree of openness'. Given *n* seat-winning parties, the 'least open' open system is one in which $S_1 + S_2 + S_n > S/2$, so that the two largest parties together with any other party can form a winning coalition, while the 'most open' system takes the form of the simple majority game.

Clearly the openness of a parliament to a variety of winning coalitions tends to increase as the number of seat-winning parties, as well as the effective number of assembly parties, increases and as the seat share of the leading party decreases. Thus openness can be expected to increase with the seat product of the electoral system.

While highly open systems enhance the representative character of an assembly (and also tend to make the bargaining power of parties more closely proportional to their seat shares and electoral support), they also present practical problems. The bargaining process required to form a government typically becomes more difficult and time consuming. Moreover, coalition governments, once formed, tend to become more fractious, as they contain more parties, and as such are less likely to endure. In fact, Taagepera has proposed an 'inverse square law' of cabinet durability: the expected lifetime of a parliamentary government is inversely proportional to the square of the effective number of assembly parties. More specifically and substituting according T5, Taagepera proposes the following expression:

(T8) *The expected durability* (in years) *of a parliamentary government* $G = 42/N_S^{\,2} = 42/(\mathrm{SP})^{1/3}$.

Note that this expression suggests that cabinet durability depends more directly on party fragmentation within parliament as a whole than on party fragmentation within the governing coalition itself.[63]

Expression T8 suggests that there is a fairly direct tradeoff between the degree to which an electoral system promotes the representative character of a parliament (increasing with the seat product) and the degree to which it promotes effective and accountable government (decreasing with the seat product). At the same time, it may also be noted that, given an assembly of typical size (on the order of $S = 250$, as assumed in Table 16), a proportional electoral rule used in relatively small (e.g., 6-12) magnitude districts, or with a relatively high (3-5%) national legal threshold, can provide good representation of all but quite small parties; at the same time, such a rule implies an effective

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[62] Note that, contrary to what the names may suggest, a top-two system is more open than a top-three system.

[63] Taagepera derives this law by the same logic (which also applies to the time it takes to form the governing coalition in the first place) pertaining to communications channels used to derive the cube root law of assembly size. The constant 42 is estimated from statistical analysis of empirical data over many electoral systems and extended time periods. The last column in Table 16 shows expected cabinet durability for varying district magnitudes (but recall that parliamentary systems with MMD majoritarian electoral rules do not actually exist).

number of parties of about 3 to 4 with the leading party probably controlling more than one-third of the seats; this in turn implies a less than a fully open system and quite likely a strongly dominant party (though almost certainly short of a one-party majority government), providing a reasonable prospect of a stable and effective coalition government.

7.3 Coalition Government Formation with Office-Seeking Parties

The preceding section dealt with coalition *possibilities* — that is, what winning coalitions *can* form in a multi-party assembly. In this section and the next, we address the question of which of the possible winning coalitions may *actually* form and on what terms. However, we address this question only schematically, in terms of two distinctly opposed assumptions about the motivation of political parties and their leaders.

In this section, we use the assumption that parties are purely *office-seeking* — that is, motivated only to win office and enjoy the perquisites thereof. Given a two-party system, this implies that the sole aim of each of the parties is to win elections and thereby hold office. (This is the standard assumption used in Module 10.) Given a multi-party system, this implies that each party (of whatever size) seeks to maximize its expected share of a 'prize' of fixed size, namely the 'spoils' of office holding. More specifically, each party aims to enter a governing coalition with the largest possible share of ministerial portfolios and other government offices. Thus the success of a party depends on whether it enters a governing coalition and its share of offices in that government.

A commonsensical expectation might be that parties in a prospective governing coalition will divide the prize in proportion to the resources each contributes to winning the prize — specifically, the seats that each party contributes to the coalition's winning status. This implies that only minimal winning coalitions form, since by definition at least one member of a non-minimal winning coalition can be expelled without jeopardizing its winning status while increasing the share of the prize for the remaining members. But, more specifically, this implies that the *minimum* (or 'cheapest') *winning coalition* — that is, the winning coalition that controls the *smallest* number of seats — forms.[64] Thus, given the seat distributions in columns (1)-(3) in Table 17, the coalition $\{P_1,P_4\}$ forms, splitting the prize 28/51 to 23/51, 46/57 to 11/57, and 48/51 to 3/51, respectively; given the seat distribution in column (4), the coalition $\{P_2,P_3\}$ forms, splitting the prize 27/51 to 24/51.

But this commonsensical expectation does not survive closer analysis. In the example given in column (4) of Table 17, P_1's expectation of sharing the prize in proportion to its (largest) seat share cannot be rationally sustained. A demand by P_1 for such a share induces P_2 and P_3 to prefer to form a coalition with each other, so P_1 gets nothing at all. Therefore P_1 must scale down its demands to the point that it is as desirable a coalition partner as the others. The end result of such a bargaining process is that any one of the three coalitions $\{P_1,P_2\}$, $\{P_1,P_3\}$, and $\{P_2,P_3\}$ may form, dividing the spoils equally between the two partners, and thereby leaving each party indifferent as to which coalition it joins.

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[64] While two or more coalitions may occasionally be tied in this status, a minimum winning coalition is typically unique.

Likewise, consider the seat distribution given in column (1) of Table 17. The expectation that coalition $\{P_1,P_4\}$ forms is sustained only because P_2 and P_3 are asking too much. Moreover P_4 is getting too much, because P_1's expectation of sharing the prize in proportion to its seat share in this case underestimates its bargaining power, as P_2, P_3, and P_4 are in effect competing with each for the opportunity to join P_1 in a winning coalition. However, there is a lower limit to this competition, because P_2, P_3, and P_4 can form their own winning coalition and split the prize equally. Thus, if P_1 enters a winning coalition with one of the other parties, it can secure two-thirds of the prize but no more. It can be checked that the same logic holds with the seat distributions given in columns (2) and (3), precisely because (as we saw in the previous section) the coalition possibilities are precisely the same.

It appears that, given office-seeking parties, any minimal winning coalition may form and its members share the prize in a way that is proportional, not to their seat shares, but to their minimal integer weights; therefore, while parties clearly want to enter winning coalitions, they are ultimately indifferent as to which minimal winning coalition they enter.[65] Since (in the absence of a majority party), every parliamentary seat configuration implies that there are least three minimal winning coalitions, the office-seeking assumption implies a high degree of indeterminacy in government coalition formation and suggests nothing about the policies the governing coalition will pursue, so the process is entirely unrelated to the preferences of the electorate. We therefore turn to the other distinctly opposed assumption about the motivations of political parties and their leaders.

7.4 Coalition Government Formation with Policy-Seeking Parties

We now turn to the more productive assumption that parties and their leaders are *policy-seeking*, i.e., motivated to enact preferred policies (e.g., as enunciated in their election manifestos). Given a two-party system, the office-seeking and policy-seeking assumptions have broadly similar implications, since parties must win elections in order to enact preferred policies. However, given a multi-party system, they have quite different implications for coalition formation: while office-seeking parties seek to enter coalitions giving them the greatest share of offices, policy-seeking parties seek to enter governing coalitions that will enact policies that correspond as closely as possible to party preferences.

Once again, our presentation will be schematic and simplified. We assume that parties are strongly *ordered* over a single policy or ideological dimension, e.g., from left to right. (This assumption is often, though not always, plausible.) We now label the parties $P_1, P_2, \ldots, P_n$, not in terms of their seats shares, but in terms of their ideological positions, so that P_1 is the most left-wing party and P_n is the most right-wing party. We call P_1 and P_n the *extreme* parties and say that P_j is *between* P_h and P_k if $h < j < k$. Of special significance is the *median* party, defined as the party P_m such that $S_1 + \ldots + S_{m-1} < S/2$ and at the same time $S_{m+1} + \ldots + S_n < S/2$ — that is, the parties to its left collectively control fewer than half of the seats, and likewise for the parties to its right. Provided

65 However, the situation is less clear when the parliamentary seat configuration produces a non-homogeneous weight majority game (which can occur only with six or more non-dummy parties). Furthermore, with a still larger number of parties, the assignment of MIWs may not be unique.

that the number of seats is odd, a unique median party always exists. A *connected* coalition is a set of ideologically adjacent parties; if two parties belong to a connected coalition, so do all parties between them. A connected coalition can therefore be specified by its most left-wing and most right-wing members, and it includes all parties between them (its *interior* members).

Given the policy-seeking assumption, each party seeks to enter into coalition with parties that are ideologically 'adjacent' to itself and therefore have somewhat similar policy goals. For example, P_k prefers P_{k-1} and P_{k+1} as coalition partners over more ideologically remote partners. This suggests that governing coalitions will be connected. A *minimal connected winning* (MCW) coalition is a coalition that is both winning and connected and contains no subcoalition (subset) that is both winning and connected. An MCW coalition that loses either of its extreme members remains connected but is no longer winning. An MCW coalition that loses an interior member may or may not remain winning but is no longer connected; the latter possibility illustrates the point that an MCW coalition may not be minimal winning — indeed, it may include a dummy.

We can state a number of essentially self-evident propositions concerning the median party and MCW coalitions.

(C1) *If there is a majority party, it is the median party and* $\{P_m\}$ *is the only MCW coalition.*

(C2) However*, a median party always exists and need not be the largest party* — indeed, it may be the smallest seat-winning party.

(C3) *The median party, and only the median party, belongs to every MCW coalition*. This follows because every winning coalition excluding the median party is disconnected.

(C4) *In the absence of a majority party, at least two MCW coalitions exist*. The coalition $\{P_1, \ldots, P_m\}$ is certainly connected and winning and, if it is not MCW, P_1, then P_2, etc., can be dropped in turn until the coalition becomes MCW; call this the *leftist* MCW. The *rightist* MCW is defined in parallel manner. In addition, there may be one or several relatively *centrist* MCW coalitions, in which P_m is an interior rather than extreme member, and from which both extreme parties are excluded.

(C5) *If* $n(P_i)$ *is the number of MCW coalitions to which* P_i *belongs, it follows that* $n(P_i) \leq n(P_j)$ *if* $i < j < m$ *and* $n(P_i) \geq n(P_j)$ *if* $m < i < j$.

As an example, consider the seat distribution in the Italian parliament following the 1963 election shown in Table 6. Table 18 shows the parties ordered from left to right, together with their seats. It can be checked that, given this distribution of seats, Christian Democracy is the median party and there are three MCW coalitions — a centrist such coalition as well as the leftist and rightist ones. Christian Democracy is also the largest party — indeed, it is strongly dominant — giving it an especially advantageous position in coalition bargaining. What is perhaps especially noteworthy is that the Communist Party, despite being the second largest party, is not needed to make the leftist connected coalition winning and therefore belongs to no MCW coalition.

The proposition that parliamentary governments are typically based on MCW coalitions receives considerable empirical support. In the era of multi-party Italian politics extending from the post-war period to the early 1990s, the great majority of governments were in fact based on MCW coalitions.[66] More generally, the hypothesis that parliamentary government coalitions are based on MCW coalitions is broadly supported by empirical evidence over many countries and time periods.

Given that some MCW governing coalition is likely to form, the coalition preferences of the median party become especially influential, since its membership in the coalition is indispensable. Therefore, largely independent of its size, the policy preferences of the median party, are likely to be the dominant influence on the policies enacted by the governing coalition.[67] In particular, the median party has an incentive to form a centrist MCW governing coalition if the arithmetic allows — in particular, one in which it is also the median member of the governing coalition — so that the policy preferences of the more left-wing and more right-wing members of the coalition counter-balance one another and allow the median party more easily to get its way most of the time. But even in a distinctly leftist or rightist MCW coalition, government policy is likely to be skewed in the direction of the median party in the assembly, rather than the median party within the coalition (if they are different), since the former belongs to, and the latter is excluded from, at least one other MCW coalition.

These considerations suggest that multi-party parliaments elected by PR tend to produce governing coalitions that are more or less centrist in terms of the distribution of seats in parliament. Since the distribution of seats closely reflects the distribution of first preferences with respect to parties (and presumably party policies) in the electorate, such coalitions are likely correspond closely to the 'center' of public opinion. In contrast, single-party governments elected by FPTP systems (and in which Duverger's Law is operating to good approximation) are likely to pursue policies that are somewhat to the left or right of the center of public opinion (though this tendency typically moderated checked by the competitive pressures examined in Module 10). But this also implies that government policy in FPTP systems is likely to be more responsive to (relatively small shifts in) public opinion, as a relatively high swing ratio translates small shifts in votes into larger shifts in seats and a probable change in the governing party. In contrast, government policy in PR systems is typically more stable, as the swing ratio is always essentially one. The most substantial shifts are likely to come when there is a fundamental realignment in the governing coalition, but this may or may not be related to substantial shifts in the party seat and vote shares and in underling public opinion. Indeed, government policy may respond *negatively* to a shift in vote and seat shares. Suppose a centrist MCW governing coalition is in place and a new election shifts seats shares considerably to the left (or right), with the result that the former coalition is no longer winning; while

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66 Previous post-WWII elections, which produced generally similar seat shares, led to the formation of right-leaning MCW governing coalitions. The 1963 election produced an 'opening to the left' in which the Socialist Party broke off its alliance with the Communist Party and Christian Democracy formed the leftist MCW coalition including the Socialists. The Communist Party was not included in any governing coalition in this era; while this exclusion is usually attributed to its 'pariah' status (especially in the eyes of Christian Democrats) in the Cold War period, the arithmetical fact is that it *never* belonged to any MCW coalition.

67 This observation — together to following observation referring to the median party within the governing coalition — reflects the logic of the 'median voter theorem' discussed in Module 6.

a new more leftist (or rightist) government may be formed, the old coalition may attempt to restore its winning status by including additional parties to its right (or left).

One implication of these considerations is that elections under PR rarely produce wholesale alternation in office, in which one governing coalition is replaced a new and wholly disjoint one. In contrast, elections under FPTP regularly produce such wholesale replacement of governments. Furthermore, while PR produces good legislative proportionality, in terms of the relationship between seats and votes, both following particular elections and also averaged over many elections, it characteristically does not produce good 'executive proportionality', in terms of the relationship between controlling ministries and either seats or votes, even over the long haul, as the same relatively centrist parties may dominate governing coalitions over extended periods. Indeed, the wholesale replacement characteristic of FPTP may produce better long-run executive proportionality .[68] though this in a sense an artifact of the operation of Duverger's Law.

As noted above, given a profile of seat shares, the median party may be small or large. A any size. If small, its median status may be fragile and lost after the next election. If large, its median status is likely to be retained over an extended perriod.A small party median party that median statusOf course, if party Pk occupies the median following a given election but has only a small seat share, But may be seat is small,

Thus PR tends to produce a higher degree of 'median correspondence' than FPTP systems.

As noted below, coalitionwise election inversion can occur under even the purest types of list PR, but they appear to be infrequent and limited in scope. Inversions of considerable frequency and scope do occur under many types of impure PR; in fact manipulation of the details of PR systems to produce inversions may be as commonplace as manipulation of SMD boundaries to produce inversions under FPTP (as discussed in Module 8).

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[68] In so far as the parliamentary government formation process produces minimal winning coalitions — or, in any case, less than 'grand' coalitions including all or almost all parties — we cannot expect executive proportionality except over the long run.

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Tables

Party	*Vote Share*	*Ideal Quota*	*Possible Apportionments of M = 5 Seats*				
			(1)	(2)	(3)	(4)	(5)
P_1	63%	3.150	5	4	3	3	2
P_2	28%	1.400	0	1	2	1	2
P_3	9%	0.450	0	0	0	1	1
Total	100%	5.000	5	5	5	5	5
Produced by Apportionment Formula			--	D'H	St.-L	LR-H	--
Measure of Dispro-portion-ality	D_1	0.000	0.370	0.170	0.120	0.110	0.230
	D_2	0.000	0.334	0.147	0.108	0.098	0.199
	D_3	0.000	0.330	0.159	0.143	0.159	0.270
	D_4	0.000	1.587	1.270	1.429	2.222	2.222

Table 11. Possible Apportionments with Three Parties and *M* = 5

Quota type ($M = 5$)	*Party*	Party A	Party B	Party C	Party D	Party E	Total
	Votes	48,537	29,084	14,029	7,475	1,112	100,237
	Vote share	0.4842	0.2902	0.1400	0.0746	0.0111	1.0000
	Ideal quota	2.4211	1.4507	0.6998	0.3729	0.0555	5.0000
$Q_H =$ 20,048	Quotas contained	2	1	0	0	0	3
	Remainders	8,441	9,036	14,029	7,475	1,112	40,093
	Seats	2	2	1	0	0	5
$Q_D =$ 16,707	Quotas contained	2	1	0	0	0	3
	Remainders	15,123	12,377	14,029	7,475	1,112	50,116
	Seats	3	1	1	0	0	5
$Q_I =$ 14,320	Quotas contained	3	1	0	0	0	4
	Remainders	5,577	14,764	14,029	7,475	1,112	42,957
	Seats	3	2	0	0	0	5

Table 12. Examples of Quota Methods of Apportionment with $M = 5$

Divisor / Seats	Party A	Party B	Party C	Party C	Party D
1	44,382 (1)	21,462 (3)	17,682 (4)	9,475 (8)	8,382
2	22,191 (2)	10,731 (7)	8,841	4,738	
3	14,749 (5)	7,151			
4	11,096 (6)				
5	8,876 (9)				
Seats for $M = 2$	2	0	0	0	0
Seats for $M = 3$	2	1	0	0	0
Seats for $M = 4$	2	1	1	0	0
Seats for $M = 5$	3	1	1	0	0
Seats for $M = 6$	4	1	1	0	0
Seats for $M = 7$	4	2	1	0	0
Seats for $M = 8$	4	2	1	1	0
Seats for $M = 9$	5	2	1	1	0

Table 13A. Application of D'Hondt Apportionment Method with $M = 2$ through 9

Formula	Divisor / Seats	Party A	Party B	Party C	Party D	Party E
Sainte-Laguë	1	44,382 (1)	21,462 (2)	17,682 (3)	9,475 (5)	8,382 (7)
	3	14,749 (4)	7,151(8)	5,894	3,158	2,794
	5	8,876 (6)	4,292			
	7	6,340 (9)				
	Seats for $M = 2$	1	1	0	0	0
	Seats for $M = 3$	1	1	1	0	0
	Seats for $M = 4$	2	1	1	1	0
	Seats for $M = 5$	2	1	1	1	0
	Seats for $M = 6$	3	1	1	1	0
	Seats for $M = 7$	3	1	1	1	1
	Seats for $M = 8$	3	2	1	1	1
	Seats for $M = 9$	4	2	1	1	1
Modified Sainte-Laguë	1.4	31,701 (1)	15,330 (2)	12,630 (4)	6,767 (7)	5,987 (9)
	3	14,749 (3)	7,151(6)	5,894	3,158	2,794
	5	8,876 (5)	4,292			
	7	6,340 (8)				
	Seats for $M = 2$	1	1	0	0	0
	Seats for $M = 3$	2	1	0	0	0
	Seats for $M = 4$	2	1	1	0	0
	Seats for $M = 5$	3	1	1	0	0
	Seats for $M = 6$	3	2	1	0	0
	Seats for $M = 7$	3	2	1	1	0
	Seats for $M = 8$	4	2	1	1	0
	Seats for $M = 9$	4	2	1	1	1

Table 13B. Application of Sainte-Laguë and Modified Sainte-Laguë Apportionment Methods with M = 5 through 9

District #	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0%	−1.5%	−40%	−45%	−12.0%	−36.0%	−30%
2	0%	−1.0%	−39%	−38%	−6.0%	−18.0%	−25%
3	0%	−0.7%	−38%	−32%	−2.5%	−7.5%	−20%
4	0%	−0.5%	−37%	−26%	−1.5%	−1.5%	−5%
5	0%	−0.3%	−36%	−20%	−0.8%	−2.4%	2%
6	0%	−0.2%	−35%	−12%	−0.4%	−1.2%	3%
7	0%	−0.1%	−34%	−6%	−0.15%	−0.45%	4%
8	0%	0.0%	0%	0%	0.0%	0.0%	5%
9	0%	0.1%	34%	6%	0.15%	0.45%	6%
10	0%	0.2%	35%	12%	0.4%	1.2%	7%
11	0%	0.3%	36%	20%	0.8%	2.4%	8%
12	0%	0.5%	37%	26%	1.5%	4.5%	9%
13	0%	0.7%	38%	32%	2.5%	7.5%	10%
14	0%	1.0%	39%	38%	6.0%	18.0%	12%
15	0%	1.5%	40%	45%	12.0%	36.0%	15%
Mean	0%	0.0%	0%	0%	0.0%	0.0%	0%
Median	0%	0.0%	0%	0%	0.0%	0.0%	5%
SD	0%	0.74%	35.8%	25.9%	5.0%	15.0%	13.4%

Table 14A. Seven Hypothetical Distributions of 15 Districts by Partisan Deviations for P_1

National Vote for P_1	*Number of Seats Won by Party P_1*						
v_1	(1)	(2)	(3)	(4)	(5)	(6)	(7)
15%	0	0	4	2	0	1	0
20%	0	0	6.5	3	0	1	0
25%	0	0	7	4	0	1	0
30%	0	0	7	4.5	0	1	0
35%	0	0	7	5	0	2	0.5
40%	0	0	7	6	1	2	2.5
44%	0	0	7	6.5	1.5	3	6.5
46%	0	0	7	7	2	4	8.5
48%	0	0	7	7	3	5	10.5
49%	0	1.5	7	7	4	6	11
50%	7.5	7.5	7.5	7.5	7.5	7.5	11
51%	15	13.5	8	8	11	9	11
52%	15	15	8	8	12	10	11
54%	15	15	8	8	13	11	11
56%	15	15	8	8.5	13.5	12	11
60%	15	15	8	9	14	13	12
65%	15	15	8	10	15	13	12
70%	15	15	8	11	15	14	12.5
75%	15	15	8	12	15	14	13.5
80%	15	15	8.5	12.5	15	14	14.5
85%	15	15	11	13	15	14	15

Table 14B. Votes-Seats Relationships (Cumulative Distributions) from Table 14A

Party	*vote share*	*squared*	*seats*	*seat share*	*squared*
Conservative Party	.361	.1303	306	.471	.2216
Labour Party	.290	.0841	258	.397	.1575
Liberal Democrats	.230	.0529	57	.088	.0077
UK Independence Party	.031	.0009	0	.000	.0000
British National Party	.019	.0003	0	.000	.0000
Scottish Nationalist Party	.017	.0003	6	.009	.0001
Green Party	.010	.0001	1	.002	.0000
Democratic Unionist Party	.006	.0000	8	.012	.0002
Sinn Féin	.006	.0000	5	.008	.0001
Plaid Cymru	.006	.0000	3	.005	.0000
Social Dem. & Labour Party	.004	.0000	3	.005	.0000
Alliance	.001	.0000	1	.002	.0000
Other	.019	.0000	2*	.003	.0000
	1.000	.2833	650	1.000	.3865
Effective Number		$N_v = 3.53$			$N_s = 2.59$

* The Speaker (whose election is by tradition not contested) plus one independent.

Table 15 — Effective Number of British Political Parties Following the 2010 General Election

Electoral Rule	*District Magni-tude*	*Seat-Product (S = 250)*	*Number of Seat-Winning Parties*	*Seat Share of Leading Party*	*Effective Number of Assembly Parties*	*Effective Number of Electoral Parties*	*Vote Share of Leading Party*	*Cabinet Duration (in years)*
Maj.	250	1	1.00	1.000	1.00	1.59	.707	42.0
Maj.	50	5	1.50	.818	1.31	1.84	.633	24.6
Maj.	25	10	1.78	.750	1.47	1.98	.600	19.5
Maj.	10	25	2.24	.669	1.71	2.19	.556	14.4
Maj.	5	50	2.66	.613	1.92	2.37	.523	11.4
Maj./Prop	1	250	3.98	.501	2.51	2.91	.448	6.7
Prop.	5	1250	5.95	.410	3.28	3.64	.379	3.9
Prop	10	2500	7.07	.376	3.68	4.02	.352	3.1
Prop.	25	6250	8.89	.335	4.29	4.61	.318	2.3
Prop.	50	12500	10.57	.308	4.82	5.12	.294	1.8
Prop.	250	625000	15.81	.251	6.30	6.56	.244	0.5

Table 15. Taagepera's Seat-Product Quantitative Relationships Illustrated with S = 250

	Seat Share Profiles			
Party	(1)	(2)	(3)	(4)
P_1	28%	46%	48%	34%
P_2	25%	25%	45%	27%
P_3	24%	18%	4%	24%
P_4	23%	11%	3%	15%
Total	100%	100%	100%	100%

Table 17. Four Possible Seat Share Profiles for Four Parties

Party	*Party Name*	*Seats*	*MCW Coalitions*		
P_1	Communist Party	166			
P_2	Socialist Party	87	X		
P_3	Democratic Socialist Party	33	X	X	
P_4	Republican Party	6	X	X	
P_5	Christian Democracy	260	X	X	X
P_6	Liberal Party	39		X	X
P_7	Monarchist Party	8			X
P_8	MSI (neo-fascist)	27			X
	Independents and others	4			
Total		630	389	338	334

Table 18. Minimal Connected Winning Coalitions in Italy, 1963

Figures

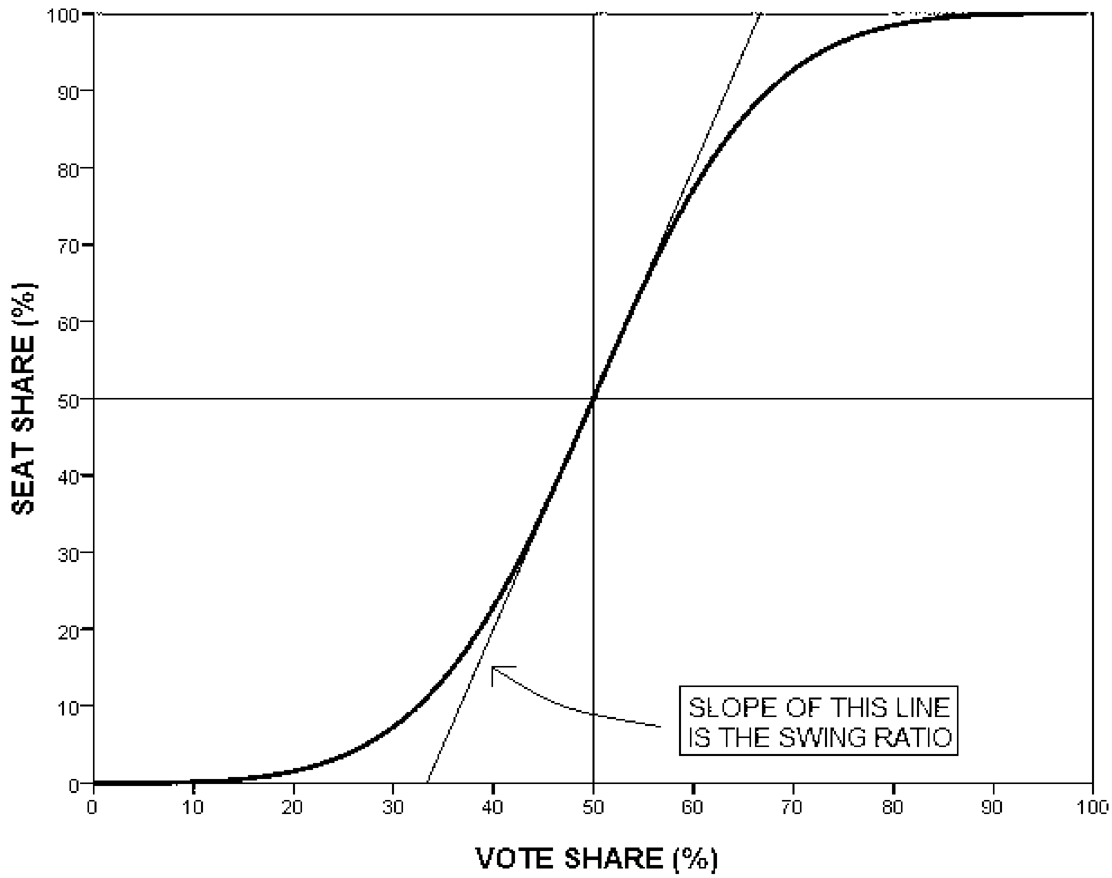

Figure 1 THE CUBE LAW

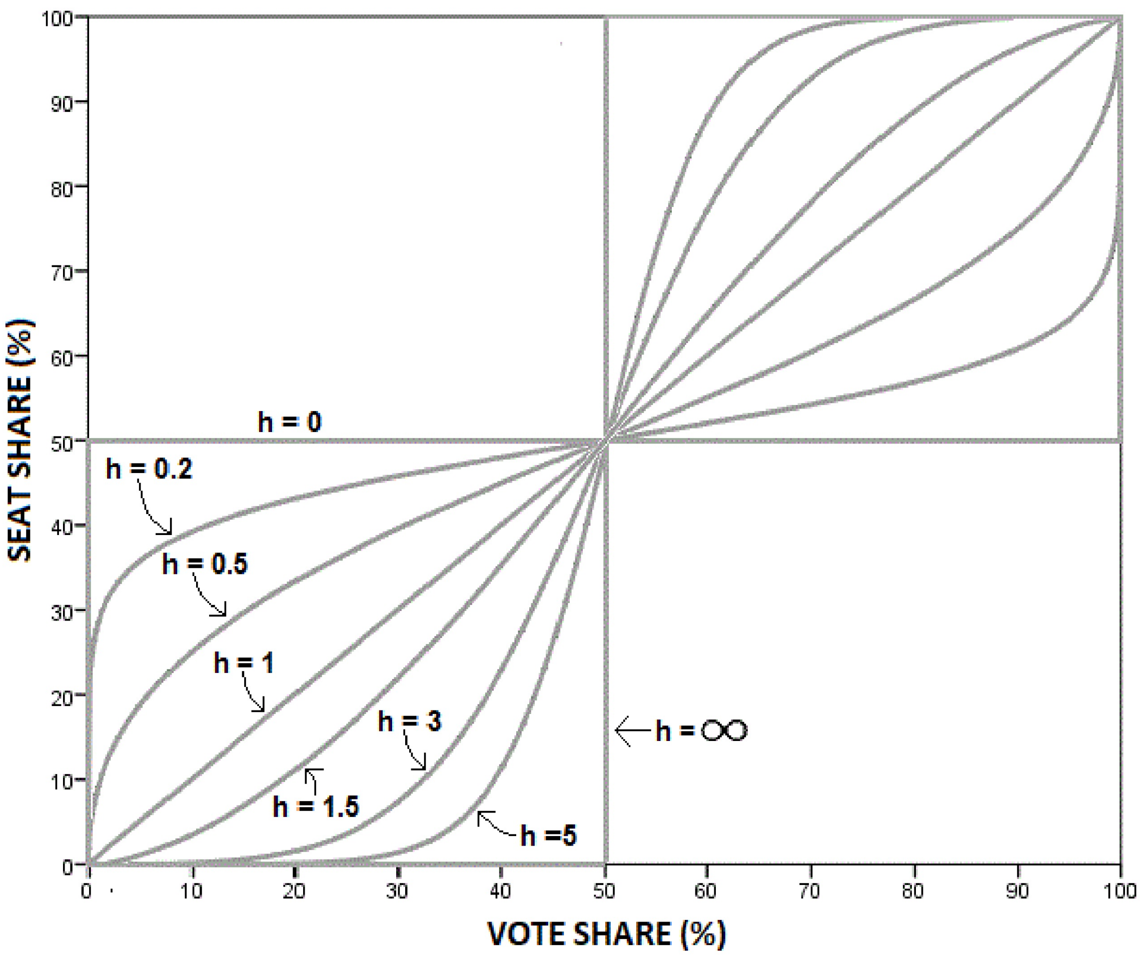

Figure 2 VOTES AND SEAT UNDER VARYING POWER LAWS

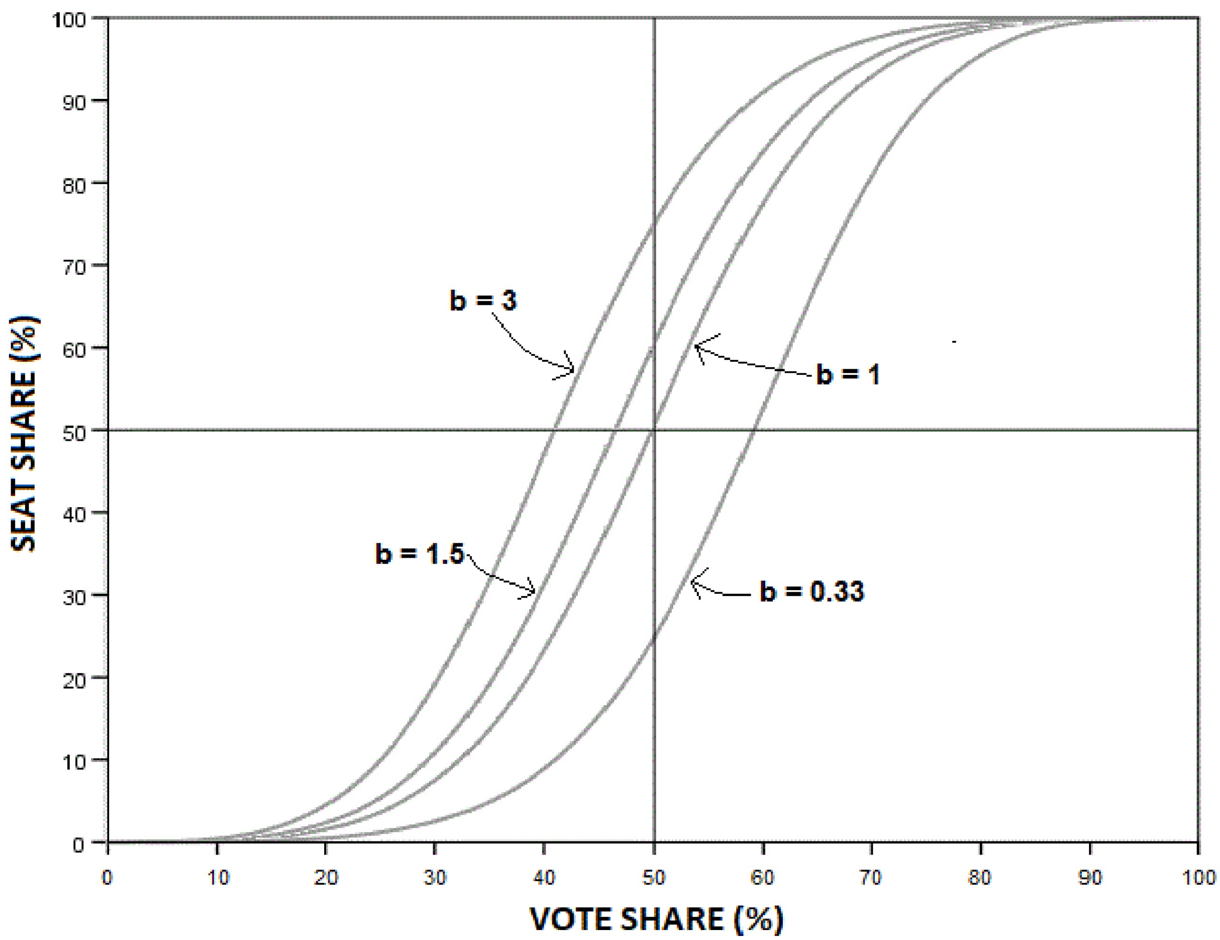

Figure 3 VOTE AND SEATS UNDER BIASED VARIANT OF THE CUBE LAW

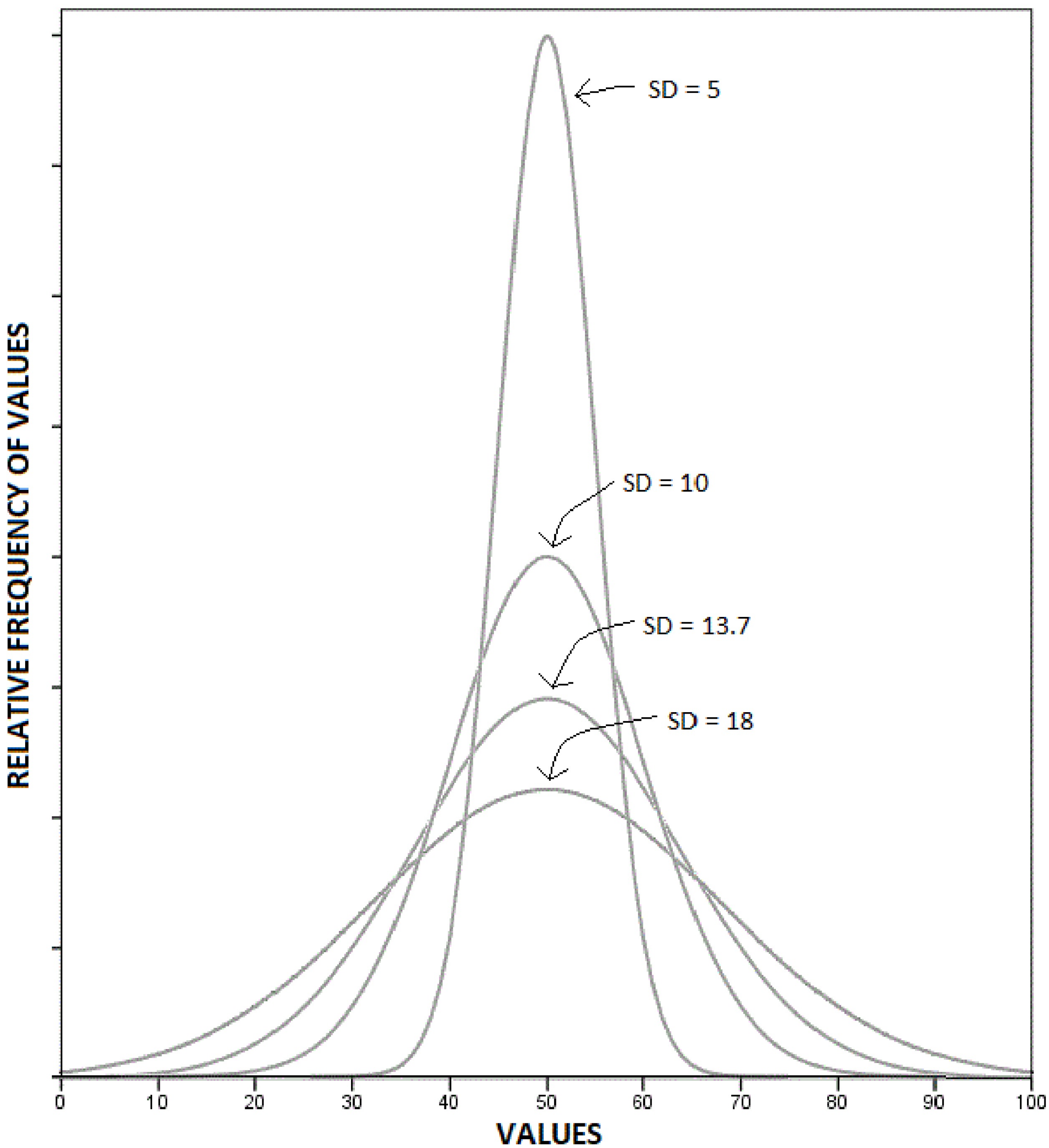

Figure 4A FOUR NORMAL DISTRIBUTIONS WITH VARYING STANDARD DEVIATIONS

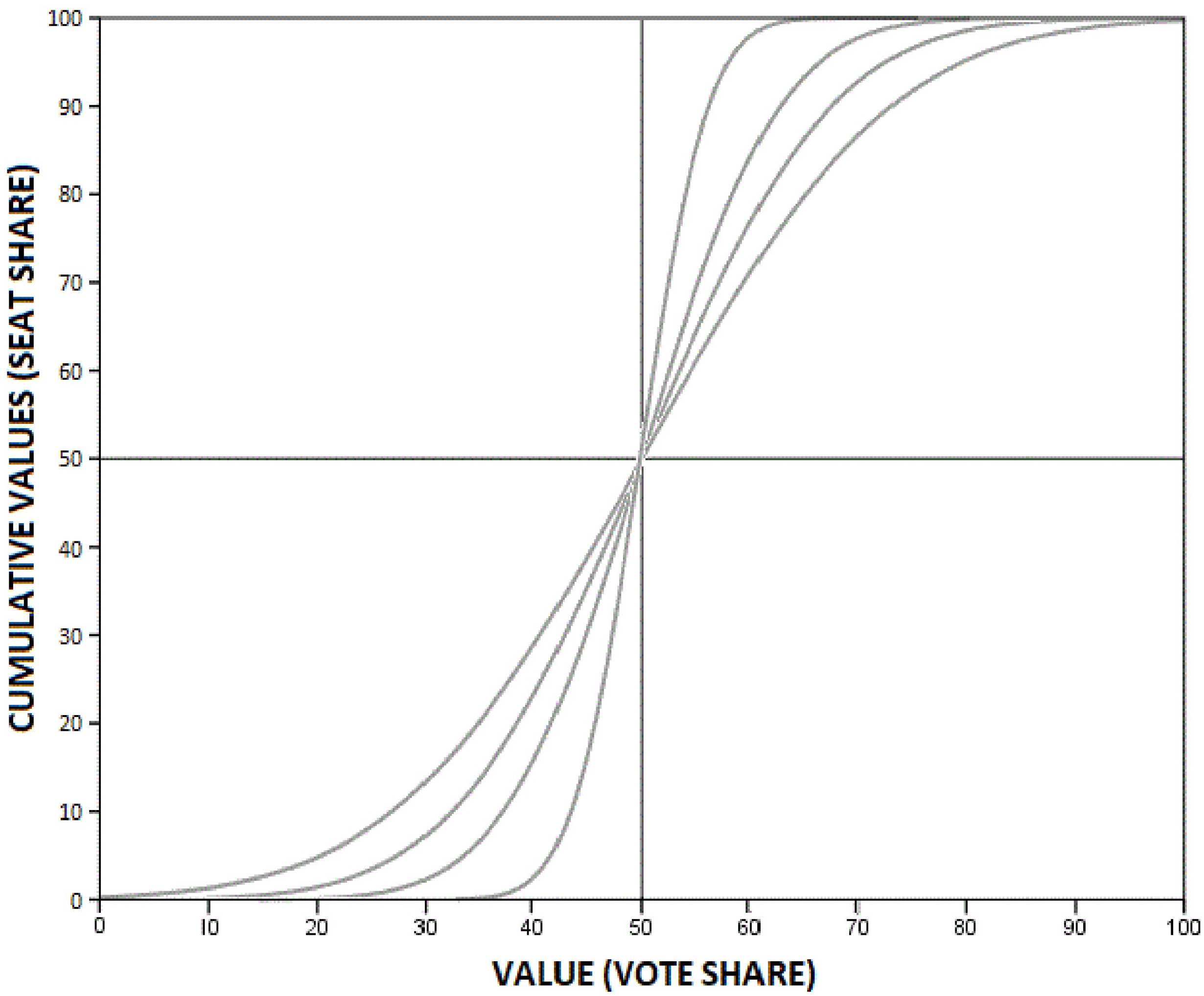

Figure 4B FOUR CUMULATIVE NORMAL DISTRIBUTIONS WITH VARYING STANDARD DEVIATIONS

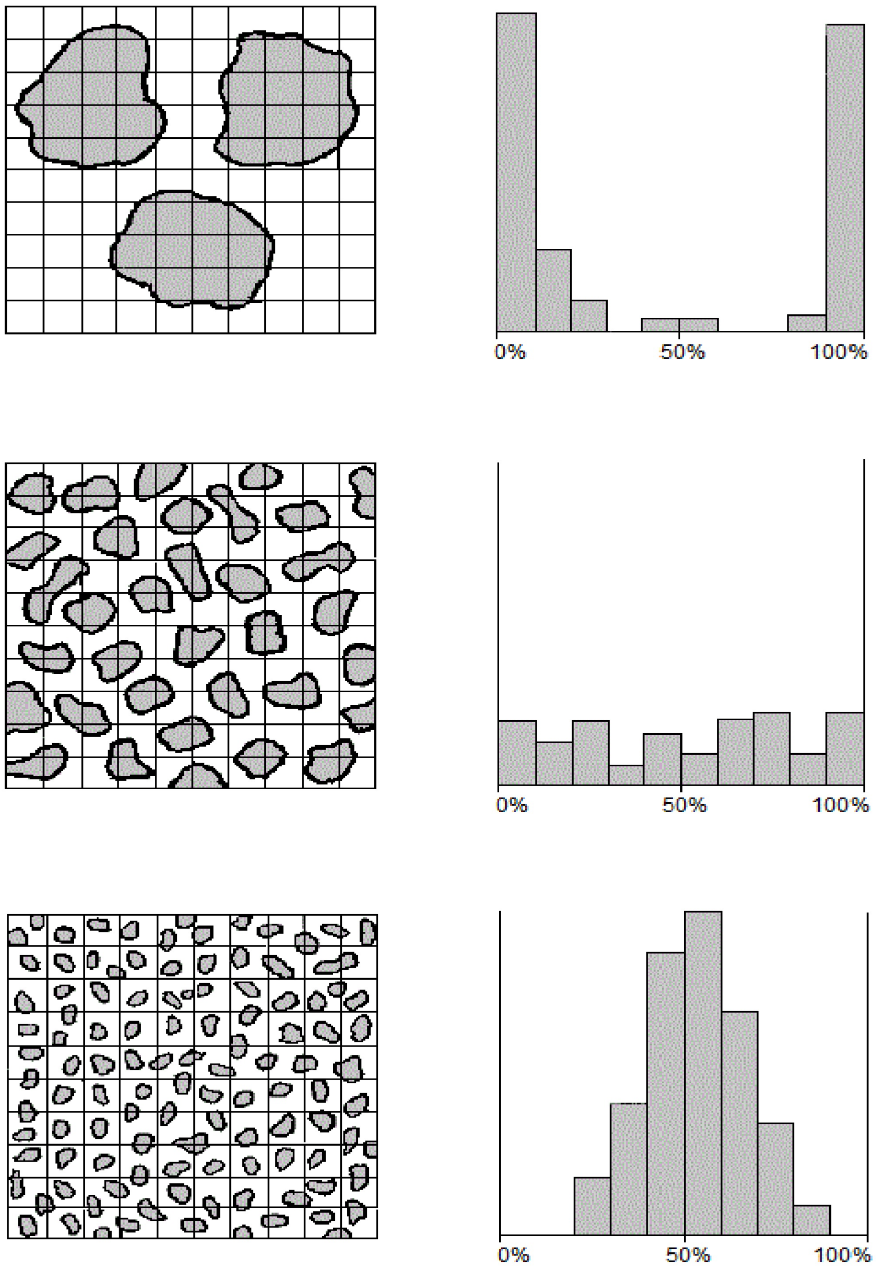

Fig. 6 PARTISAN CLUSTERS AND DISTRICT SIZE

Source: Gudgin and Taylor (1979)

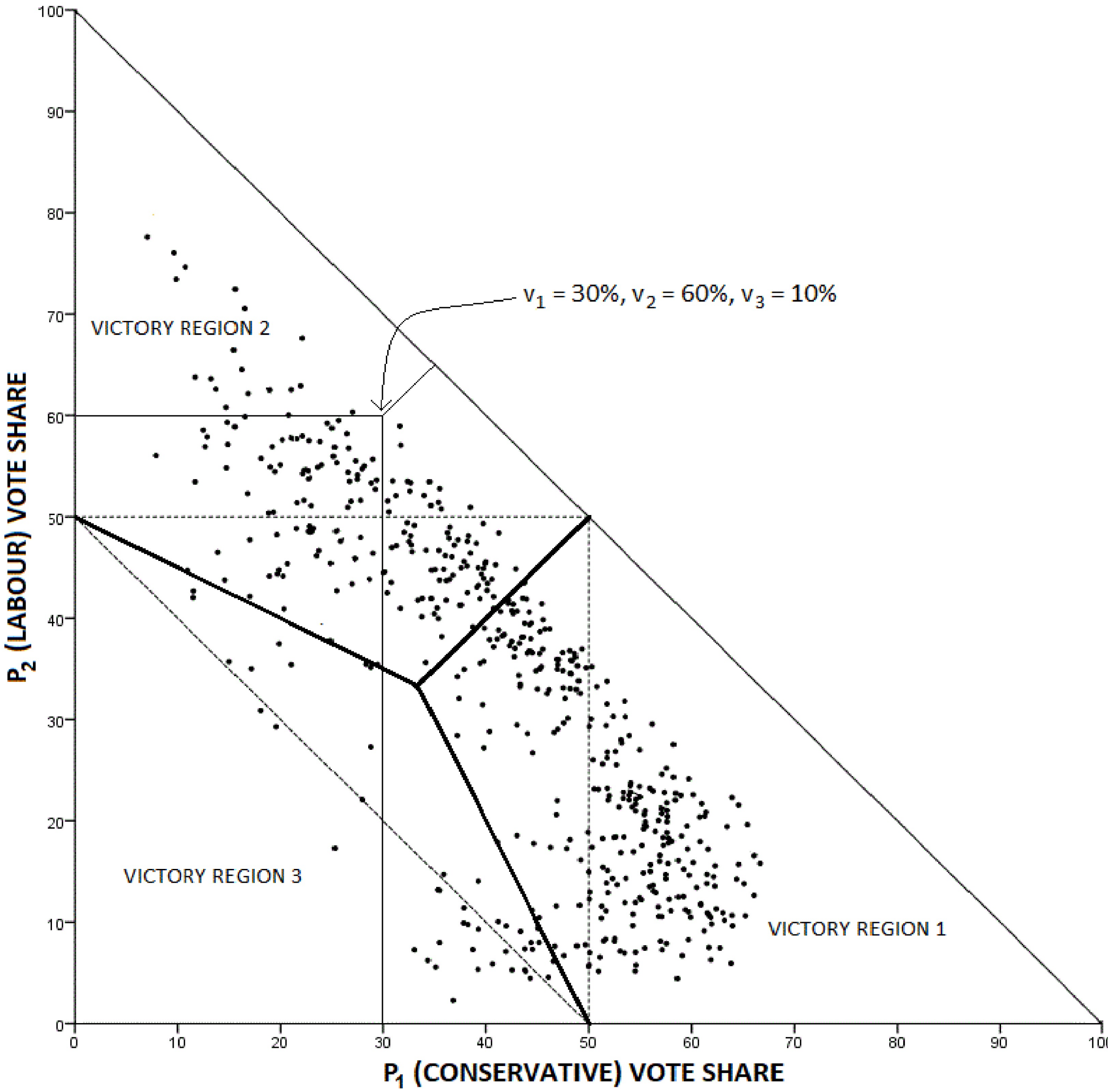

Figure 7 AN THREE-PARTY ELECTION TRIANGLE: ENGLAND IN 2010

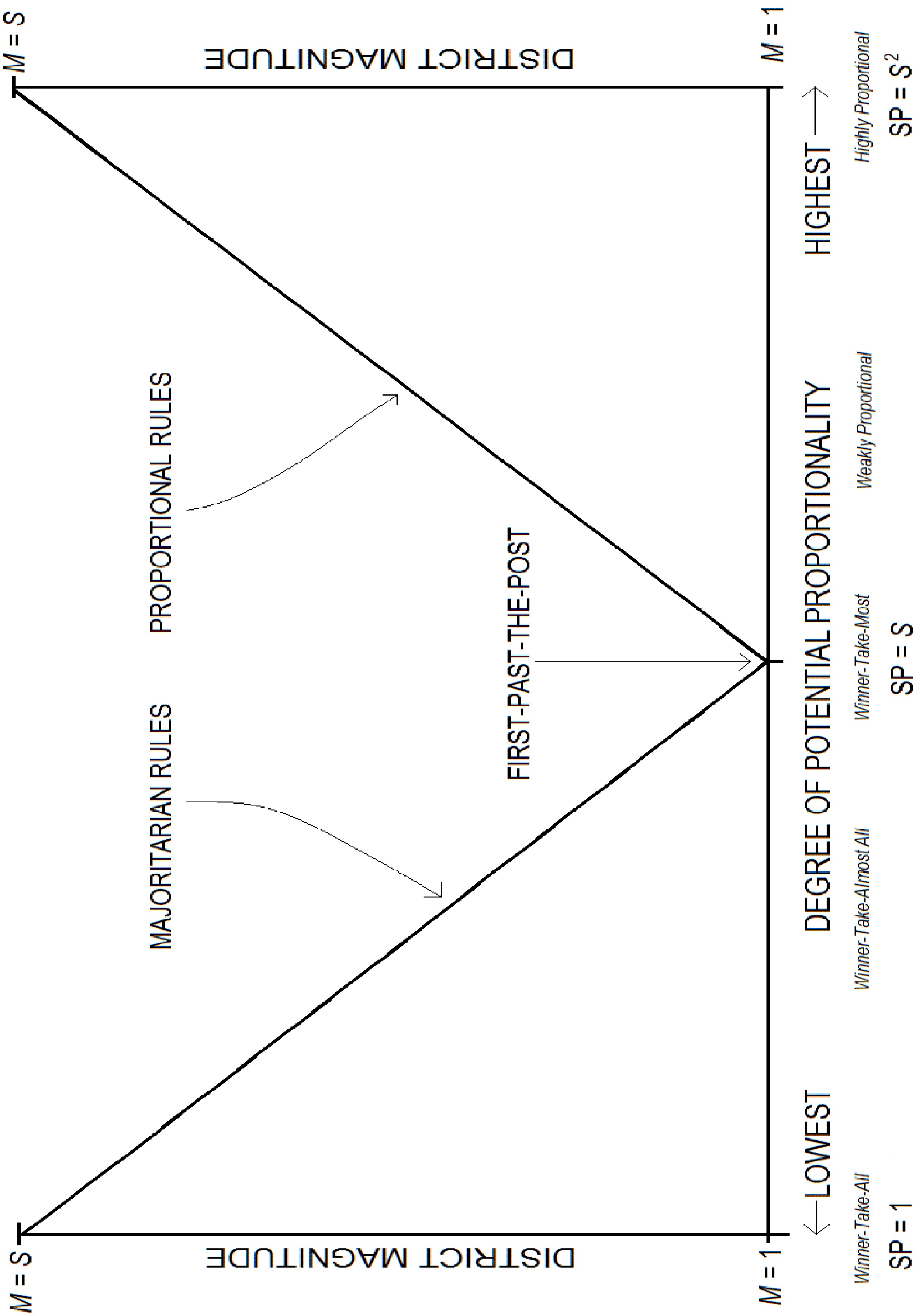

Figure 7 PROPORTIONALITY OF ELECTORAL SYSTEMS BY DISTRICT MAGNITUDE