# THE GEOMETRY OF VOTING CYCLES: THEORETICAL DEVELOPMENTS

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## Abstract

The fact that political choice by majority rule may lead to voting cycles has been the subject of ongoing scholarly concern since it was rediscovered some fifty years ago by Duncan Black and Kenneth Arrow. However, part of this research has revealed an apparent anomaly: while a variety of theoretical results suggest that they are pervasive, it has been relatively difficult to turn up empirical evidence of voting cycles, even where appropriate data — that is, data on individual preference orderings over three (or more) candidates, parties, or legislative alternatives — are available. In this paper, I address this apparent discrepancy between formal theory and common empirical findings. In the first section, I briefly review some relevant theory and the empirical findings, introducing some notation, terminology, concepts along the way, and then observe there is no *logical* tension between the theoretical and empirical results. In the remaining sections, I go beyond this observation to show theoretically that only in quite exceptional circumstance can a voting cycle exist among three alternatives embedded in a space of two (or more) dimensions, where voters have preferences based on Euclidean distance. In doing this, I exploit a simple geometrical construction suggested by recent empirical and theoretical work.

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The fact that political choice by majority rule may lead to voting cycles has been the subject of ongoing scholarly concern since it was rediscovered some fifty years ago by Duncan Black (1948, 1958) and Kenneth Arrow (1951). However, part of this research has revealed an apparent anomaly: while a variety of theoretical results suggest that they are pervasive, it has been relatively difficult to turn up empirical evidence of voting cycles, even where appropriate data — that is, data on individual preference orderings over three (or more) candidates, parties, or legislative alternatives — are available.

In this paper, I address this apparent discrepancy between formal theory and common empirical findings. In the first section, I briefly review some relevant theory and the empirical findings, introducing some notation, terminology, concepts along the way, and then observe there is no *logical* tension between the theoretical and empirical results. In the remaining sections, I go beyond this observation to show theoretically that only in quite exceptional circumstance can a voting cycle exist among three alternatives embedded in a space of two (or more) dimensions, where voters have preferences based on Euclidean distance. In doing this, I exploit a simple geometrical construction recently introduced by Adams and Adams (2000) in a more empirically oriented paper.

## 1. Theoretical and Empirical Background

Suppose we have a *one-dimensional* ideological or policy space and a set of voters with preferences over the space. Suppose further that these preferences are *single-peaked*, i.e., that each voter has an *ideal point* (point of highest preference) somewhere along the dimension and that his satisfaction with alternatives declines with distance away from his ideal point. We say that point $X$ *is majority preferred to* another point $Y$ (for which we write $X \rightarrow Y$ and sometimes say "$X$ beats $Y$") if the number of voters who prefer $X$ to $Y$ exceeds the number who prefer $Y$ to $X$. A *voting cycle* is a sequence of alternatives such as $X \rightarrow Y \rightarrow Z \rightarrow X$, i.e., such that majority preference "cycles back" and repeats itself. Duncan Black's *Median Voter Theorem* (1948, 1958) tells us that, in the one-dimensional case with single-peaked preferences, *no* voting cycles of any length can occur: that is, given any two points $X$ and $Y$ such that $X \rightarrow Y$, there are no other points $P_1, \ldots, P_k$ such that $Y \rightarrow P_1 \rightarrow \ldots \rightarrow P_k \rightarrow X$. This implies that there is at least one point not beaten by any other point. Indeed, if the number $n$ of voters is odd, it implies that there is a unique point (called the *Condorcet winner*) that beats every other point, which coincides with the ideal point of the median voter when voters are ordered by the locations of their ideal points.

If in the world of empirical political choice alternatives were typically arrayed over a single commonly perceived ideological or policy spectrum, the paucity of observed voting cycles would be predicted by Black's theorem. But a good deal of empirical research suggests that the ideological or policy space over which candidates and parties compete for electoral support in the U.S. and elsewhere has (at least) *two* dimensions. Studies of contemporary American mass public opinion strongly indicate that the economic/welfare and cultural/national dimensions of opinion are independent and largely uncorrelated in the mass public (see in particular Shafer and Claggett, 1995). Other empirical work (e.g., Budge et al., eds, 1987; Schofield, 1995; Schofield et al., 1997; Lijphart, 1998) likewise finds that two dimensions are necessary (and substantially sufficient) to represent voter preferences over multiple competing parties in recent national elections in a wide variety of countries.

Social choice theory has produced two well-known results about the behavior of majority rule when the number of dimensions increases beyond one. First, if the ideological or policy space is expanded to two (or more) dimensions (while voter preferences remain single-peaked over all straight lines through this space), voting cycles can easily occur. Indeed, Charles Plott's *Majority Rule (Dis)Equilibrium Theorem* (1967), together with related results,[1] tells us that the "radial symmetry" condition on voter preferences necessary for the existence of a Condorcet winner in such a space are extremely stringent and almost always fails to hold. Second, Richard McKelvey's *Global Cycling Theorem* (1976, 1979) tells us that, in the almost certain event that the symmetry condition fails to hold and there is no Condorcet winner, voting cycles are *pervasive* and encompass *all* points is the space.

Thus, McKelvey's Theorem, in conjunction with the empirical findings previously noted, might seem to imply that voter preferences over candidates, parties, bills, or other alternatives located in an ideological space of two or more dimensions will typically exhibit cycles. While scholars have (arguably) identified a number of example of voting cycles in parliamentary or legislative committee settings where the (effective) number of voters is small, the (relatively few) surveys or other data sources (e.g., ordinal ballots available for scholarly analysis) that elicit information (or allow inferences) about the preference *orderings* over *three* (or somewhat more) alternatives held by relatively large numbers of voters rarely turn up evidence of voting cycles.[2]

This paper shows that in fact there is no tension between McKelvey's theoretical result and the empirical findings noted here. More specifically, we use a simple geometrical construction to show that, given *an electorate that is even moderately large* and even in the face of the McKelvey Theorem, voter preferences over any *three alternatives* located in *two-dimensional space* are very unlikely to exhibit voting cycles.[3]

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[1] In particular, Enelow and Hinich (1983); also see Cox (1987). Plott's theorem was anticipated in earlier work by Black and Newing (1951).

[2] For example, Feld and Grofman (1992) examine ordinal ballots from 36 STV elections in British and Irish professional associations. Among the 15,599 triples of candidates, only 71 (0.4%) resulted in voting cycles. Further analysis on a more inclusive version of the same data set is provided by Felsental et al. (1993) and Felsenthal and Machover (1995). Similar research includes Niemi (1970), Dobra and Tullock (1981), Dobra (1983), Chamberlin, et al.(1984), Niemi and Wright (1987), Abramson et al. (1995), Brams and Merrill (1994), Radcliff (1997), Regenwetter and Grofman (1998), Van Deeman and Vergunst (1998), and Adams and Adams (2000). In such data, all logically possible preference orderings actually occur, so the voters do not commonly perceive the alternatives as arrayed across a single dimension.

[3] Tangian (2000) reach a somewhat similar conclusion by a quite different logical route.

As a first step, it is useful to specify exactly what the McKelvey Theorem does, and does not, say. This is what the McKelvey theorem *does* say: in the almost certain absence of a Condorcet winner, if we take any *two* arbitrary points $X$ and $Y$ in a two-dimensional space such that $X \rightarrow Y$, we can always find a finite set of other points $P_1, \ldots, P_k$ such that $Y \rightarrow P_1 \rightarrow \ldots \rightarrow P_k \rightarrow X$, so that $X$ and $Y$ appear in a common voting cycle. Consequently, any point in the space can be reached from any other point by following a path of majority preference.

But the McKelvey Theorem does *not* say this: if we take any *two* arbitrary points $X$ and $Y$ in the space such that $X \rightarrow Y$, we can always find some *single* third point $P$ such that there is a *three-element* cycle $X \rightarrow Y \rightarrow P \rightarrow X$.[4] And even less does the McKelvey Theorem say this: if we take any *three* arbitrary points $X$, $Y$, and $Z$ the space, we can expect to find a 3-cycle that includes *precisely* these three alternatives. The upshot is that there is certainly no *logical contradiction* between the McKelvey Theorem and the empirical findings noted earlier. This is by no means a novel claim,[5] but I do believe that it has rather often been overlooked in discussions of this problem. What I show in the remainder of the paper — and what I do believe is new — is this: only in rare and peculiar circumstances will we find a voting cycle encompassing three alternatives arbitrarily located in a two-dimensional space.

The remainder of the paper is laid out as follows. I first present the geometrical construction due to Adams and Adams (2000) that allows us to apply various social choice ideas due to Sen (1966), Niemi (1969), and Feld and Grofman (1986) to voter preferences in a two-dimensional space. In terms of this construction, I then identify conditions on the location of the alternatives and the distribution of ideal points that determine whether majority preference exhibits a cycle over the three alternatives, and I show that these conditions are likely to be satisfied only rarely. Finally, I link these social choice ideas with concepts in the spatial theory of voting — in particular the "yolk" — to identify several necessary conditions for a voting cycle among three alternatives to exist and again observe that they are quite stringent.

For convenience of exposition, I consistently refer to the alternatives that are the object of social choice as "candidates," though obviously the analysis pertains just as well to choice among parties, legislative options, etc. For ease of analysis, I assume that there are just *three candidates*, that candidate locations and voter ideal points are embedded in a *two-dimensional* space, and that voter preferences are *Euclidean* — that is, each voter has an ideal point in the two-dimensional space and prefers any alternative closer to his ideal points to a more distant one (so voter indifference curves are concentric circles).[6] I briefly consider the generalizability and limitations of these conclusions in the concluding discussion.

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4 That is, the shortest majority preference path cycling back to $X$ may require many alternatives. (It is true that every point in the space belongs to a 3-cycle, so there is a 3-cycle including $X$ and a 3-cycle including $Y$, but this does not imply that there is a 3-cycle including both $X$ and $Y$.)

5 The point was made explicitly and emphatically by Richard Niemi (1983, p. 261). The following analysis strongly supports the more general claim that Niemi was making.

6 The earlier (less general) version of the Global Cycling Theorem presented by McKelvey (1976) specifically applies in this context.

## 2. The Candidate Triangle and Preference Profiles

Suppose we have a two-dimensional space and a distribution of *n* voter ideal points over that space (where *n* is either odd or so large that ties effectively never occur). Suppose also that three candidates locate at points in the same space. Given Euclidean preferences, each voter prefers a candidate who is closer to his ideal point to one who is more distant. We want to investigate the conditions under which this setup entails a voting cycle among the candidates.

If the three candidates are located at the same point, all voters are indifferent among them, so there is no voting cycle. If the three candidates are located along a straight line, the dimensionality of the space is effectively reduced to one and voter preferences are single peaked, so there is no cycle. We therefore assume that the three candidate locations *X*, *Y*, and *Z* are *distinct* and *non-collinear*, defining the vertices of what we shall call the *candidate triangle*.

Now let us construct the perpendicular bisector of the *XY* side of the candidate triangle, i.e., the locus of points equidistant from *X* and *Y*, and also do the same for the other two sides. A basic theorem of plane geometry tells us that the three bisectors intersect at a common point that is equidistant from the three vertices, which we call the *hub* of the candidate triangle. Thus the three bisectors jointly partition the space into six "pie-slice" *sectors* separated by six "spokes" emanating from the hub. We call this geometrical construction the *preference partition* resulting from the candidate triangle.[7] (See Figures 1-3 for illustrative partitions resulting from candidate triangles of different shapes.)

Given Euclidean preferences, every voter whose ideal point lies on the *X* side of the perpendicular bisector of the *XY* side prefers *X* to *Y*, while every voter whose ideal point lies on the *Y* side prefers *Y* to *X*.[8] Voter preferences between *X* and *Z* and also *Y* and *Z* likewise depend on which side of the *XZ* and *YZ* bisectors ideal points lie. Thus all voters whose ideal points lie in a given sector have identical preference orderings over *X*, *Y*, and *Z* — the nature of the common ordering being determined by the side of each bisector the sector lies on. Figures 1-3 display the preference orderings associated with each sector and labels the sectors and orderings in the following manner.

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[7] This preference partition superimposed on the distribution of voter ideal points over a two-dimensional space is the geometrical construction introduced by Adams and Adams (2000). However the more general representation of preferences ordering over three alternatives as six "slices" of a wheel or triangle goes back at least to Saari (1994) and Lissowski and Swistak !995). Also see Regenwetter, Adams, and Grofman (2000) and Tabarrok (2001). Note that this construction can also represent the six weak orderings which express indifference between two candidates (i.e., voters whose ideal points lie exactly on one of the six "spokes") and the one "null ordering" expressing indifference among all three candidates (i.e., a voter whose ideal point lies exactly on the hub).

[8] Thus the perpendicular bisector is the two-dimensional generalization of what is commonly called the "cut point" in one-dimensional voting models. It is convenient, and not particularly restrictive, to assume that no voter ideal point lies precisely on any bisector, and we carry forth the discussion in the main text accordingly.

| ***Sector*** => | ***X1*** | ***Y2*** | ***Z1*** | ***X2*** | ***Y1*** | ***Z2*** |
|---|---|---|---|---|---|---|
| *Highest preference* | *Z* | *Z* | *Y* | *Y* | *X* | *X* |
| *Medium preference* | *X* | *Y* | *Z* | *X* | *Y* | *Z* |
| *Lowest preference* | *Y* | *X* | *X* | *Z* | *Z* | *Y* |

We make the following observations. Geometrically opposite sectors are associated with opposite preference orderings over *X*, *Y*, and *Z*. Since opposite orderings have the same medium preferences, each ordering (and the associated sector) is labeled *X*, *Y*, *Z* according to its *medium* ranked element. Sectors are alternately labeled *Type 1* or *Type 2*, so opposite sectors are of opposite types. Voters in alternating adjacent sectors share the same first preference and the same last preference.

If we now superimpose a preference partition over the actual distribution of voter ideal points, we can count up the number of ideal points in each sector to determine its *population*, i.e., *n*(*X1*), *n*(*X2*), where the six sector populations add up to *n*. A list *n*(*X1*), . . . , *n*(*Z2*) is a voter *preference profile*, which specifies the number of voters with each possible preference ordering of the candidates.[9]

Each preference profile produces a particular majority preference pattern over the alternatives *X*, *Y*, and *Z*. Apart from the possibility of ties in majority preference, there are eight possible majority preference patterns over the three alternatives: the six orderings *X1*, . . . , *Z2* listed above, plus two cyclical patterns commonly referred to as the *forward cycle* ($X \rightarrow Y \rightarrow Z \rightarrow X$) and the *backward cycle* ($X \rightarrow Z \rightarrow Y \rightarrow X$).[10]

Social choice theory has identified a number conditions on preference profiles over three alternatives that are sufficient to preclude the two cyclical patterns of majority preference. Here we state such conditions using the the framework and terminology of the geometrical preference partition displayed in Figures 1-3.

The conditions are of two types. *Popularity conditions* pertain to the the proportion of voters that have particular orderings. *Strong population dominance* holds if *more than half the ideal points lie in a single sector* (i.e., a majority of voters have the same ordering), in which case majority preference simply the dominant ordering, so a cycle is precluded regardless of the relative popularities of the remaining sectors. *Weak population dominance* holds if *more than half the ideal points lie in two adjacent sectors*. In terms of preference orderings, this means that there is one

[9] Note that, provided the candidate locations form a triangle (i.e., are distinct and non-collinear), voter ideal points may be distributed in such a way that all logically possible preference orderings of the three candidates occur in the electorate — indeed, provided that the number of voters is reasonably large and that candidates are not more extreme ideologically than the most extreme voters, it is essentially guaranteed that all preference orderings occur in the electorate

[10] Ties can result if the number of voters *n* is even or (regardless of whether *n* is even or odd) if some voters are indifferent between some candidates. Allowing for ties (i.e., "majority indifference") means that majority rule may additionally produce the seven weak orderings noted in footnote 6 plus six "quasi-transitive" and six "acyclical" patterns. (See Sen and Pattanaik, 1969).

candidate who is the highest preference of a majority of voters or who is the lowest preference of a majority of voters, in either event precluding a cycle.

*Exclusion* (or *value-restriction*) *conditions* stipulate that certain sectors be entirely unpopulated (i.e., excluded) but stipulate nothing about the relative popularity of the non-excluded sectors. Exclusion conditions sufficient to preclude voting cycles were set out comprehensively by Sen (1966). First we note that, if only Type 1 sectors are populated and population dominance does *not* hold (i.e., no Type 1 sector includes a majority of ideal points), the forward cycle $X \to Y \to Z \to X$ results; likewise if only Type 2 sectors are populated and population dominance does *not* hold, the backward cycle $X \to Z \to Y \to X$ results.

More importantly, it can be verified that a voting cycle can result *only if* all three alternating sectors (either Type 1 or Type 2) are populated. To see this, note that every (cyclical or non-cyclical) majority preference pattern must include a candidate who both beats and is beaten by another candidate, e.g., $Y$ where $X \to Y \to Z$. By definition $X \to Y$ if and only if

$$n(X1) + n(Z2) + n(Y1) > n(X2) + n(Z1) + n(Y2)$$

and $Y \to Z$ if and only if

$$n(Y1) + n(X2) + n(Z1) > n(Y2) + n(X1) + n(Z2).$$

If $X \to Y$ and $Y \to Z$, both inequalities hold, from which it follows that $n(Y1) > n(Y2)$. Since $n(Y2) \geq 0$, it further follows that $n(Y1) > 0$. Thus the forward cycle $X \to Y \to Z \to X$ can occur only if

$$n(X1) > 0 \text{ and } n(Y1) > 0 \text{ and } n(Z1) > 0$$

and the backward cycle $X \to Z \to Y \to X$ can occur only if

$$n(X2) > 0 \text{ and } n(Y2) > 0 \text{ and } n(Z2) > 0.$$

Putting the matter the other way around, a voting cycle is precluded if at least one sector of each type is unpopulated. This can result if (i) two adjacent sectors are unpopulated or (ii) two opposite sectors are unpopulated. Condition (i) is called *single-peakedness* in the event the two adjacent unpopulated sectors share the same *last* preference. In this event, the populated sectors are "value-restricted" in that there is a candidate that no voter ranks last, so that candidate beats both other candidates unless the population dominance condition comes into play, and in either case a voting cycle is precluded. Condition (i) is called *single-cavedness* in the event the two adjacent unpopulated sectors share the same *first* preference. In this event, there is a candidate that no voter ranks first, so that candidate is beaten by both other candidates unless the population dominance condition comes into play, and in either case a voting cycle is precluded. Condition (ii) may be called *polarization*. In this event, there is a candidate that no voter ranks *medium*. Since opposite sectors are unpopulated, there are two pairs of adjacent (possibly) populated sectors, one pair of which must include a majority of ideal points, so weak population dominance holds.[11]

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[11] Note that "single peakedness" in this context is defined specifically with respect to preferences over a triple of alternatives, not over all points on a line. Though we are assuming that all voter preferences over the two-dimensional space are Euclidean (a generalization of single-peakedness in the latter sense), it is quite possible for voter preferences over three (non-

Each of these conditions is *sufficient* by itself to preclude a voting cycle. But even the conjunction of population dominance and an exclusion condition fails to be *necessary* to preclude a cycle. Moreover, given even a modestly large number of voters, it is unlikely that *any* sector of the preference partition will be *entirely* unpopulated,[12] especially if (as seems plausible) the candidate triangle is more or less centrally located relative to the distribution of ideal points. However, many years ago Niemi (1969) showed that orderings need not be entirely excluded to preclude a voting cycle, provided some orderings are *relatively* unpopular. Subsequently, Feld and Grofman (1986) pushed Niemi's idea to the limit by developing the concept of "net preferences."

Clearly a profile in which *two opposite orderings* have the *same popularity* produces the same majority preference pattern as the *reduced* profile that excludes both of these orderings, as voters with the two opposite orderings cancel each other out at every pairwise choice. More generally, any profile $n(X1)$, . . . , $n(Z2)$ produces the same majority preference pattern as the *net preference profile* $n'(X1)$, . . . , $n'(Z2)$ in which only the "surplus" orderings in each of the three pairs of opposite orderings remain. That is, if $n(X1) \geq n(X2)$ in the original profile, $n'(X1) = n(X1) - n(X2)$ and $n'(X2) = 0$ in the net preference profile, and likewise for the *Y* and *Z* sectors.[13] We call the sum of these surpluses the *net population*, composed of "net voters."

We can now apply *net population dominance* and *net exclusion* conditions to a net preference profile to state a *necessary and sufficient* condition to preclude a voting cycle.

In a *net preference partition*, the opposite of a net populated sector must be net unpopulated. Thus at most three sectors can be occupied by net voter ideal points, and the possible combinations of such net populated sectors are distinctly limited. Thus there are just two structural possibilities for a net preferences partition: (i) a *fragmented* pattern, in which no bisector has all net populated sectors to one side of it (and three alternating sectors are net populated); and (ii) a *compact* pattern, in which all net populated sectors lie on the same side of some bisector (so that, if there are three net populated sectors, they are adjacent). There are two possible fragmented patterns: all Type 1 sectors or all Type 2 sectors are net populated, which — in the absence of net population dominance, i.e., unless one sector includes a majority of the net population — produce the forward and backward cycles respectively. There are six possible compact patterns in which three adjacent sectors are populated, in which case majority preference corresponds to the ordering associated with the middle sector, unless net population dominance comes into play, and in any case a voting cycle over *X*, *Y*, and *Z* is precluded.[14]

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collinear) candidate locations to be polarized or single-caved.

[12] This observation is congruent with more formal and general results set out by Kramer, 1973.

[13] Feld and Grofman (1986) say that $n'(X2) = n(X2) - n(X1)$, i.e., the net popularity of an ordering is simply the negative of its opposite. This is neater mathematically but may less clearly capture the intuition of preferences "canceling out" that underlies the net preference construction. In any case, the difference is merely a matter of presentation.

[14] A net preference partition is unlikely to have fewer than three populated sectors is if the number of voters is large, because this requires that the (gross) population of two opposite sectors be *exactly* equal. But if there are fewer than three populated sectors, net population dominance assures that a cycle is precluded.

To summarize, a preference profile over three candidates results in a voting cycle if and only if the corresponding net preference partition exhibits a fragmented structure and no sector has a majority of the net population.

## 3. The Preference Partition and Voting Cycles

The presence or absence of a voting cycle among the three candidates *X*, *Y*, and *Z* depends on the population — more particularly, on the net population — of the six sectors of the preference partition. The (net) population of the six sectors is in turn determined jointly by three factors: the *shape of the candidate triangle* (i.e., the locations of the candidates relative to one another), the *location of the candidate triangle* (relative to the distribution of voter ideal points), and the *distribution of voter ideal points*.[15]

With respect to the *shape* of the candidate triangle, we have three broad possibilities. The candidate triangle may be (approximately) equilateral (as in Figure 1), in which case its hub is at (or near) the center of the triangle and all sectors are defined by the same (approximately) 60° angle and, in that sense, all are (approximately) the same "size." As the shape of the candidate triangle is deformed from equilateral perfection, the angles defining the sectors change, but opposite sectors necessarily have the same angle.

If the candidate triangle is deformed in the obtuse direction by pulling two "extremist" candidate locations (say *X* and Y) apart and/or pushing one "centrist" candidate location (say Z) toward the *XY* line, the hub is pulled outside and away from the triangle on its long *XY* side, and the angle of the *Y* pair of sectors widens (approaching 180°), while the angles of the other pairs narrow, resulting in a configuration like Figure 2.

If the candidate triangle is deformed in the opposite fashion by pushing the locations of two "clone" candidate (say *Y* and *Z*) towards each other and/or pulling the location of a third "distinctive" candidate (say *X*) away from the other two, the hub remains within the candidate triangle but the angles of *X* pair of sectors narrow while the two others expand (each approaching 90°), resulting in a configuration like Figure 3.

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Feld and Grofman refer to such "net value restriction" as "partial single peakedness" (referring to the original, not the reduced, profile) As noted, a net preference profile includes at most three orderings. A net profile that includes all Type 1 orderings or all Type 2 orderings produces a cycle unless the net population dominance condition is met. Otherwise, a net profile either: (a) includes fewer than three orderings, or (b) includes three orderings, at least one of each type. If (a), the net profile trivially meets all value restriction conditions simultaneously. If (b), the net profile includes one ordering from each pair of opposites. Thus every alternative is ranked medium in some ordering. But since the three orderings cannot all be of the same type, one alternative must never ranked best and one never ranked worst in the net profile, which therefore is both single-peaked and single-caved. Thus, when Feld and Grofman characterize all compact net preference profiles condition single-peaked, they are correct S though such net profiles meet other value restriction conditions at the same time.

[15] It is worth observing that the *size* of the candidate triangle (i.e., the degree of "convergence" or "divergence" among the candidates) need not influence the size (or population) of the six sectors of the partition, as each set of bisectors is associated with an infinite family of similar candidate triangles of varying size and with the same hub. However, as we move to smaller triangles in the same family, they converge on the common hub. Thus, if a large triangle in a given family is distant from its hub (as in Figure 2), smaller triangles in the same family move in the direction of the common hub.

With respect to the *location* of the candidate triangle relative to the voter distribution, the most obvious consideration is whether the candidate triangle is (approximately) *centered* on the voter distribution or whether it is distinctly *off-center*. But the consideration that turns out to have decisive theoretical relevance whether the *hub* of the candidate triangle coincides (approximately) with the center of the voter distribution. The hub of an equilateral triangle coincides with its center (however defined), but in other triangles the hub and the center may diverge — most dramatically in the case of an obtuse triangle such as in Figure 2.[16]

Clearly the distribution of voter ideal points can assume a great variety of patterns. For the moment, let us suppose that the distribution is (at least approximately) bivariate normal. If voter ideal points are (approximately) equally dispersed in each dimension and substantially uncorrelated between dimensions, the ideal points form a (more or less) "circular" cloud, the density of which diminishes at the same rate in every direction from the center (i.e., the mode). Let us call this a *circular* voter distribution. To the extent that ideal points are concentrated in one dimension relative to the other and/or there a strong correlation between the two dimensions, the cloud of voter ideal points "condenses" into an *elongated* distribution, the density of which diminishes in every direction from the center but at different rates (though at the same rate in opposite directions).

In order to identify conditions that produce a voting cycle, we may imagine plotting a distribution of ideal points on a board and then placing on top of it a transparency that shows a particular candidate triangle or, more particularly, the associated preference partition. We then try to position the transparency in a way that produces a fragmented net voter partition and perhaps a voting cycle over the three candidate positions. Quick thought experiments reveal that this is very difficult to do. Indeed, given a bivariate normal voter distribution, there is no way to position the partition that clearly produces a fragmented pattern.

Let us first consider the wholly symmetric overall configuration — that is, an *equilateral* candidate triangle *centered* on the mode of a *circular* distribution of voter ideal points. In this case, each of the six sectors of the partition is (essentially) equally populated with voter ideal points, and each type of net preference partition is essentially equally likely to occur. Since there are eight types, two fragmented types which produce a voting cycle (forward and backward respectively) and six compact types which produce the six possible orderings, such a wholly symmetric configuration produces a fragmented pattern about 25% of the time. But even if a fragmented pattern results, net population dominance is likely to hold, as the overall symmetry of the configuration implies that net populations are close to zero, and it is quite likely that one net populated sector has a majority of the (relatively tiny number of) net ideal points. More precisely, we now have in effect (with respect to the three candidates) an "impartial culture" (to use a now standard term introduced by Garman and Kamien, 1969, to refer to a situation in which all individual preference orderings of three alternatives are equally likely), in which the probability of a voting cycle approaches 0.0877 (from below) as the

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16 In principle, we need a precise definitions of the "center" of both the voter distribution and the candidate triangle. If the voter distribution is bivariate normal, its mode obviously defines its center. More generally, the center of the "yolk" (discussed in the following section) defines the center of the voter distribution. We might define the center of the candidate triangle in a parallel manner, i.e., as the center of the circle that can be uniquely inscribed within the candidate triangle. Alternatively, we might define it as the overall mean of the candidate locations. (These two definition give substantially different answers in a triangle such as that displayed in Figure 3, but approximately similar ones for the triangles in Figures 1 and 2 .) But since it is the hub, not the "center," of the candidate triangle that plays the decisive theoretical role, we can sidestep this question.

number of voters becomes very large.[17] Thus, even in this most favorable circumstance, the probability of a voting cycle is quite small.

Of course, if the configuration is "exquisitely" symmetric, so that all sectors have a net population of zero, the majority preference pattern is the "null ordering" of three-way binary ties. (If we demand that *n* be odd, one sector has a net population of one and determines the majority preference pattern.) But if the symmetry is slightly less exquisite, so that the (gross) populations of all sectors are almost but not exactly equal, each pairwise choice between two candidate is almost but not quite a tie and the resulting majority preference pattern is highly fragile, in that shifts in the location of a few voter ideal points can easily shift majority preference from a cyclical to non-cyclical pattern or vice versa (and from a forward to backward cycle or vice versa or from one ordering to another). That is, whatever the majority preference pattern, it is distinctly "weak" in that each majority preference relationship is carried by a bare majority and can easily be reversed if a few voters change their preference. Thus, given an (approximately) circular bivariate normal distribution of ideal points, a voting cycle can exist among three candidates only because they virtually tie one another in pairwise contests. (In this circular case, the candidates are also virtually tied with respect to first preferences, i.e., the kind of three-way virtual tie that we might observe in a plurality election or preference poll.)

Second, let us consider an equilateral candidate triangle centered on the mode of an elongated distribution of voter ideal points. The populations of opposite sectors of the partition still must (virtually) balance, creating (virtual) ties among the three candidate in pairwise contests, and the same conclusions apply as in the circular case. (In this elongated case, however, the candidates may have quite different numbers of first preferences, so a virtual three-way tie typically would not be observed in a plurality election or poll.)

Third, we observe that shifting an equilateral candidate triangle even slightly off-center with respect to the mode of a circular or elongated distribution of ideal points produces a compact net profile, precluding a voting cycle.

Next, note that nothing in this argument rests on the fact that, given an equilateral candidate triangle, the three sectors of the partition are the same "size" (in terms their defining angles at the hub), since what matters is the balance of ideal points between opposite sectors (which are necessarily the same "size"). Thus the considerations outlined in the previous paragraphs apply to candidate triangles of any shape, with this important proviso. The previous analysis really was based based not on the location of the *center* of the candidate triangle relative to the mode of the voter distribution but on the location of the *hub* of the triangle relative to the mode of the voter distribution (though the center and the hub coincide in the equilateral case). In general, the center (however defined) and the hub of the candidate triangle rarely coincide. And except in the case of an equilateral candidate triangle (itself a somewhat improbable limiting case), there is no particular reason to expect the hub of the candidate triangle to coincide (even approximately) with the mode of the voter distribution. The more reasonable expectation (given notions of candidate competition and survival) is that the candidate triangle itself may be (approximately) centered on the mode of the

17 The probability calculations may be found in Garman and Kamien (1969), Niemi and Weisberg (1969) and elsewhere. I am indebted to James Adams and Christian List for help in developing this connection to prior results.

voter distribution. And except for the limiting equilateral case, this implies that the hub of the candidate triangle does *not* coincide with the the mode of the voter distribution but is located within some sector of the voter partition (or possibly on the boundary between two sectors), with the result that several adjacent sectors are considerably more densely populated with ideal points than their opposites. Thus, so long as the mode of the an (approximately) bivariate normal voter distribution deviates "substantially" from the hub of the candidate triangle, a compact net preference profile must result, thereby precluding a voting cycle.[18]

This conclusion applies most powerfully in the case of an obtuse candidate triangle such as that in Figure 2. We would certainly expect the mode of the voter distribution to be in the vicinity of the candidate triangle and thus distant from the hub, so the distant sectors (associated with the orderings *Y1* and *X2*) would be very sparsely populated compared with their opposites. Thus two adjacent sectors would be excluded from the net preference partition, precluding a voting cycle is precluded.[19]

The kind of candidate triangle depicted in Figure 3 leads to a similar conclusion, though admittedly less powerfully. Again, a voting cycle may occur only if the hub of the candidate triangle (virtually) coincides with the mode of the voter distribution. Since the hub now is located within the triangle, this is not an entirely implausible possibility. Still, the vicinity of the hub is a small portion of the space (and of the candidate triangle) and, if the mode of a bivariate normal voter distribution is located at any "significant" distance from the hub, a cycle is precluded. Probably the most reasonable expectation is that the mode of the voter distribution lies within the candidate triangle but somewhat in the direction of the two "clone" candidates *Y* and *Z*.[20]

To this point, considering only bivariate normal (and thus unimodal) voter distributions, we conclude that in almost circumstances a voting cycle cannot occur and, even in the most favorable limiting circumstance, there is less than one in ten chance that a voting cycle occurs and then only as a somewhat flukish result of virtual ties in the three pairwise contests.

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[18] In the next section, we give precise meaning to the proviso "substantially."

[19] In Niemi's (1969) terminology, voter preferences are "partially unidimensional." In geometrical terms, either the candidates hardly differ on one dimension (if the long and thin candidate triangle is oriented horizontally or vertically) or the candidates positions on the two dimensions are highly correlated (otherwise). In political terms, the candidates are coming close to arranging themselves along a single dimension. Of course, if the candidates arrange themselves in a perfectly collinear fashion (so that the candidate triangle has no area), the three perpendicular bisectors that define the voter partition are parallel lines causing sectors *Y*1 and *X*2 to disappear entirely. Such (nearly or fully) collinear candidate positioning induces (partial or full) single peakedness on voter preferences over the three candidates, even though voter preferences are not so structured over the whole space. This pattern of nearly unidimensional elite positions in conjunction with less structured mass preferences seems to occur rather commonly in the empirical world, as is powerfully illustrated for Britain and France in the data presented in Adams and Adams (2000) and for Germany (and, to a lesser extent the Netherlands) in data presented by Schofield et al. (1997).

[20] That is, we might expect the candidate triangle to be "centered" on the voter distribution. If we define the center to be the overall mean, that center the about 2/3 of the way from *X* to the *YZ* line, well away from the hub (which is at about the mid-point of this distance) If we defined the center of the candidate locations to be the center of the "yolk" of the triangle (see footnote 15 and the next section), it would be even closer to the *YZ* line and further from the hub.

On the other hand, if we consider "lumpy" multi-modal distributions of ideal points, we can of course configure candidate triangles that produce a fragmented net preference partitions and voting cycles. Moreover, they may be "strong"cycles in which the pairwise contests are not virtual ties but are won decisively.

However, not any multimodal distribution will do. For example, a *bimodal* distribution composed two superimposed (circular or elongated) unimodal distributions with different centers will not produce cycles. The configuration required to reliably produce a voting cycle is a candidate triangle more or less centered on a *tri-modal* voter distribution, where (i) most ideal points lie close to some mode, (ii) approximately one third of the ideal points are associated with each mode, and (iii) the modes are located within alternating sectors of the voter partition, thereby producing a fragmented pattern.[21] We may observe that this requires that the candidates locate where voters are *not* concentrated, while it would be more reasonable to expect that each candidate would locate at (or near) one of the three modes. If one candidate does locate at or near each mode and even if this results in an (approximately) equilateral candidate triangle, a voting cycle is reliably precluded unless almost exactly one third of the electorate with each node (in which case majority preference again is fragile). If the shape of the candidate triangle is deformed substantially from equilateral perfection in the obtuse direction (as in Figure 2), two candidate locations appear in adjacent sectors of the voter partition; if two modes of the voter distribution are similarly located, a voting cycle is precluded. If the shape of the candidate triangle is deformed in the manner of Figure 3, the locations of the "clone" candidates appear in adjacent sectors, while the location of the "distinctive" candidate lies close to the boundary between the opposite sectors (barely inside *Z2* in Figure 3). Unless they are extraordinarily concentrated in the neighborhood of the mode near where the the distinctive candidate locates, ideal points associated with that mode will be split between the two sectors opposite those occupied by the clones, so the distinctive candidate (decisively) loses the pairwise contest with each clone,[22] thereby precluding a cycle.

## 4. The Preference Partition and the Yolk

The preceding discussion had two important limitations. First, by speaking of the mode (or modes) of the voter distribution, we in effect assumed that the the distribution of ideal points is a continuous density (i.e., that the number of voters is infinite). Second, in so far as we focused on (approximately) bivariate normal distributions we in effect assumed that the condition of "radial symmetry" sufficient for a Condorcet winner in the space as a whole is (approximately) satisfied. In this section, we drop all assumptions about the distribution if ideal points and identify sufficient conditions for precluding a voting cycle among three candidate stated in terms of two parameters that

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[21] The most obvious theoretical possibility is to have an electorate of just three voters whose ideal points define the three modes located as described. If the number of voters at each mode increases equally (or at least sufficiently equally that population dominance does not occur), the cyclical majority preference is unchanged, as is also true even if ideal points become somewhat dispersed about the three modes.

[22] Of course, with respect to the splitting of *first preferences*, the advantage may well go the other way, and the distinctive candidate may win a plurality election against the two clones.

characterize any distribution of ideal points — namely, the location and size of the "yolk" of the distribution.[23]

A straight line *L* through the space partitions voter ideal points into three sets: those that lie on one side of *L*, those that lie on the other side of *L*, and those that lie on *L*. A *median line M* partitions the ideal points so that no more than half of them lie on either side *M*. Given that *n* is odd, fewer than half the ideal points lie on either side of *M*, and there is exactly one median line perpendicular to any line through the space.

If all median lines happen to intersect at a common point *c*, this point is the Condorcet winner in the space. If all voters have distinct ideal points, a common point of intersection requires that the distribution of ideal points be radially symmetric in the fashion required by Plott's Theorem and related results. This means that it is possible to pair off all voters but one so that their ideal points lie precisely on opposite sides of *c* and that *c* is the ideal point of the remaining voter.

While it is very unlikely that all median lines intersect in a common point, it is likely that they all pass near the center of the distribution of ideal points, so that there is a relatively small region (though not a single point) through which all median lines pass. The *yolk* is the region bounded by the circle of minimum radius that intersects every median line. The location of the yolk (specified by is its *center c*) indicates the generalized center (in the sense of the median) of the voter distribution. The size of the yolk (specified by its *radius r*) indicates the extent to which the configuration of ideal points departs from one that produces a Condorcet winner.

A continuous bivariate normal distribution (with an infinite number of voters) has a yolk of zero radius coinciding with the mode. But if the number of voters is finite, almost certainly there are slight "imperfections" in the symmetry of the distribution that prevent all median lines from intersecting precisely at a single point, so the yolk has a positive but small radius.

If there are just three voters, the three sides of the "voter triangle" are median lines and the yolk is the circle inscribed within this triangle. If the "voter triangle" is approximately equilateral, the yolk is large relative to the distribution of ideal points, though the yolk shrinks rapidly as the voter triangle deviates (in either fashion) from equilateral perfection. More importantly, the size of the yolk diminishes relative to the dispersion of ideal points as the number of ideal points increases, provided they do not cluster about distinct modes (Feld, Grofman, Miller, 1988; Tovey, 1992a 1992b). The essential point is that the proliferation of ideal points tends to make the distribution less "lumpy" and more closely approximate a continuous one. Of course, a peculiar distribution with clusters of points — in particular, a tri-modal distribution such as discussed near the end of the last section — may have a yolk that remains large relative to the dispersion of ideal points (because many median lines pass through the vicinity of two modes but not the third — put otherwise, because we are approximately back to the three-voter case) even as the number of ideal points increases.

Given a distribution of ideal points with a yolk of zero radius, point *X* in the space is majority preferred to point *Y* if and only if *X* is closer to *c* than *Y* is (Davis, DeGroot, and Hinich, 1972; Miller

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[23] This term and concept is due to McKelvey (1986), drawing on earlier work by Ferejohn, McKelvey, and Packel (1984). For a reasonably accessible discussion, from which the present discussion borrows, see Miller, Grofman, and Feld (1989).

Grofman, and Feld, 1989). That is, the *win set* of *X* (i.e., the set of points that beat *X* ) is the set of points bounded by a circle centered on *c* and passing through *X*. But if the yolk has a positive radius *r*, the boundary of the win set becomes irregular — in some places falling short of the circle and in others pushing beyond it. But there are definite bounds on this irregularity that depend on the size of the yolk. The bound that can be most readily stated is this: *Y* beats *X* only if *Y* is no more than 2*r* further from the center c of the yolk than *X* is (Miller, Grofman, and Feld, 1989). Thus we can state the following.

**PROPOSITION 1.** *A voting cycle over three candidates X, Y, and Z is precluded if the location of one candidate, say X, is either 2r closer to c, or 2r further from c, than the locations of both Y and Z.*

In this event, *X* either beats, or is beaten by, both *Y* and *Z*, and a cycle is precluded regardless of majority preference between *Y* and *Z*. The power of this proposition obviously varies inversely with the magnitude of *r*. Further the condition is only sufficient, and certainly not necessary. Indeed the 2*r* bound actually applies only if *Y* and *X* lie on opposite sides of the yolk (180° apart), though the bound is not much less if X and Y are 120° apart (as they would be if they were vertices of an equilateral candidate triangle centered on *c*). However, the bound is much tighter if *X* and *Y* lie on the same side of *c*.[24]

Probably more significant than Proposition 1 is the way the yolk relates to the perpendicular bisectors that define the preference partition.

**OBSERVATION 1.** *If the yolk lies entirely on the X side of the perpendicular bisector of XY, X → Y; if it lies entirely on the Y side of the perpendicular bisector of XY, Y → X. If the yolk straddles the bisector, we may have either X → Y or Y → X.*

There is some median line *M* perpendicular to the *XY* line. By the definition of a median line, a majority of ideal voter points lie on *M* or on the *X* side of *M*, and by the definition of the yolk, *M* must pass through the yolk. Thus if yolk lies entirely on the *X* (or *Y*) side of the perpendicular bisector, a majority of ideal points lie on the *X* side of the bisector, so *X* → *Y* (or *Y* → *X*).

The following proposition follows directly from Observation 1.

**PROPOSITION 2.** *If the yolk lies entirely within one sector of the preference partition, majority preference over X, Y, and Z is identical to the ordering associated with that sector, so a voting cycle is precluded.*

If the yolk lies entirely within some sector, it does not intersect any bisectors, so the conclusion follows from three applications of Observation 1. Note that, while majority preference is identical to the preferences of voters in the sector containing the yolk, this sector need not contain

[24] More precise bounds on the win set of *x* are given by a cardioid centered on *c* and with *x* as the focal point (Ferejohn et al., 1984; Miller, Grofman, and Feld, 1989).

(anything like) a majority of voter ideal points (i.e., Strong Population Dominance need not hold). What is true is that the sector containing the yolk and both adjacent sectors are net populated and collectively include a majority of ideal points.

However, even if the yolk is quite small and /or distant from the hub of the candidate triangle, it is still quite likely that the center of the yolk *c* lies close enough to a sector boundary that the yolk straddles two sectors. In this case (and in the absence of any information about the distribution of ideal points other than the parameters *c* and *r*), majority preference between the two candidates whose bisector is straddled is indeterminate, but a voting cycle is still precluded.

**PROPOSITION 3.** *If the yolk lies entirely within two sectors of the preference partition, a voting cycle is precluded.*

Clearly the two sectors must be adjacent. By two applications of Observation 1, either one candidate beats both of the others (if the two adjacent sectors have the same first preference) or one candidate is beaten by both of the others (if the two adjacent sectors have the same last preference).

Thus we can state the following.

**PROPOSITION 4.** *There can be a voting cycle over the three candidate locations only if the yolk of the distribution of voter ideal points intersects three or more sectors of the preference partition.*

If the yolk intersects exactly three sectors, the yolk lies entirely on one side of one bisector. Suppose it lies entirely on the *X* side of *XY*. Thus we know that $X \rightarrow Y$ but the other two majority preference relationships are indeterminate. Thus majority preference must be one of the following four patterns that include $X \rightarrow Y$: ordering *X*1, *Z*2, or *Y*1 or the forward cycle. If the yolk intersects more than three sectors (and thus includes the hub), any of the eight possible majority preference patterns may arise.

If the yolk is sufficiently small and/or the center of the yolk is sufficiently distant from the hub, the yolk cannot intersect more than two sectors of the partition. Suppose the center *c* of the yolk lies in a sector whose defining angle is $\alpha$. Let *d* be the distance from *c* to the hub. When *c* is equidistant from the two boundaries of the sector, we have the smallest possible ratio $r/d$ that allows the yolk to straddle *both* boundaries of the sector (also the largest possible ratio $r/d$ that allows the yolk to lie between both boundaries.) This critical ratio occurs when the yolk is tangent to both spokes, giving us a right triangle with hypotenuse *d*, angle $\alpha/2$, and opposite side *r*. Thus we can state the following.

**PROPOSITION 4**. *Suppose the center of the yolk is located at distance d from the hub and lies within a sector of the preference partition with a defining angle* $\alpha$. *A voting cycle is precluded if* $d < r/\sin\alpha$.

In the case of a sector with a defining angle of 60° (e.g., any sector in a partition resulting from an equilateral candidate triangle), a voting cycle is precluded if the distance from the hub to the center of the yolk *c* is twice the yolk radius *r*.

It is worth stressing that these conditions are sufficient to preclude a voting cycle among the three candidates; they are by no means necessary, and they should be considered in conjunction with the conditions identified in to preceding section. Suppose, for example, that the yolk lies between the candidate triangle and the hub and intersects sectors *Z2*, *X1*, and *Y2* in Figure 2. By the analysis in this section, there may be a voting cycle among *X*, *Y*, and *Z*. But by the analysis in the preceding section, a voting cycle can occur only if fragmented net preference partition and such a yolk does not imply a such fragmented pattern.

## 5. Concluding Discussion

The entire analysis focused on the case of *three candidates* with locations in the same *two-dimensional space* over which members of an electorate have *Euclidean preferences*. The basic argument has been that, in such a setup, it is quite unlikely that voter preferences over the three candidates will generate a voting cycle. In concluding, let us consider briefly how the arguments given in the preceding sections of the paper can be extended.

*More than two dimensions*. Suppose that voter ideal points occupy a space of three (or more) dimensions. Provided the number of candidates remain three, the previous arguments remain undisturbed. The candidate triangle defines a two-dimensional plane, the perpendicular bisectors defining the preference partition become parallel (hyper)planes perpendicular to this plane. Each sector has a population of ideal points, and we proceed exactly as before.

*More than two candidates in two dimensions*. Suppose we remain in two-dimensional space but increase the number of candidates to four (or more). Suppose (for all the reasons outlined above) that there is no voting cycle among the candidates *X, Y*, and *Z*, and suppose in particular that $X \rightarrow Y$, $X \rightarrow Z$, and $Y \rightarrow Z$. Now suppose a fourth candidate P enters the field. This raises the question of whether the cycle $X \rightarrow Y \rightarrow Z \rightarrow P \rightarrow X$ may result. But note that this can be true only if the *three-element* cycle $X \rightarrow Z \rightarrow P \rightarrow X$ exists as well as one of the three-element cycles $X \rightarrow Y \rightarrow P \rightarrow X$ and $Y \rightarrow Z \rightarrow P \rightarrow$ Y). By the previous analyses, such three-element cycles are unlikely. Given a unimodal voter distribution, such a pair of three-element cycles can occur only if the hub of *both* candidate triangles (say *XZP* and *XYP*) coincide closely with the mode of the voter distribution and thus with each other. This is logically possible but clearly less likely than the same for a single candidate triangle. And given a trimodal distribution of ideal points, an especially peculiar configuration of candidate locations would have to exist to generate both three-element cycles.

*More than two candidates in more than two dimensions*. Given four candidates located in a two-dimensional space populated by voters with Euclidean preferences, some of the 24 logically possible preference orderings of the four candidate cannot be associated with any region of the space. However, if we increase the dimensionality of the space along with the the number of candidates, all preference orderings can be associated with sectors of the space. Suppose we have four candidates in a three-dimensional space. This results a *candidate tetrahedron* with four triangular faces. Our present analysis applies to each face (triple of candidates), each of which may produce a cycle.

*Voters with non-Euclidean preferences*. The analytical technology of this paper (i.e., perpendicular bisectors [generalized cut points], median lines, the yolk, etc.), like much other spatial modeling, depends on the (vastly) simplifying assumption of Euclidean preferences. My intuition is that the general thrust of results based on Euclidean preference extends to rather more general (e.g., strictly quasi-concave) preferences.

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